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Analysis of Reaction Forces In a Beam Leaning Against a Wall

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I. Summary

Despite the ubiquity of the situation of a ladder standing against a wall, there does not yet exist a universally accepted, valid solution for the mechanical problem. In addition to being a socially relevant problem, it requires the use of interesting analytical techniques in statics and the mechanics of solids to fully address. Here, we present a model of the physics of a ladder and determine the reaction forces on the ladder.

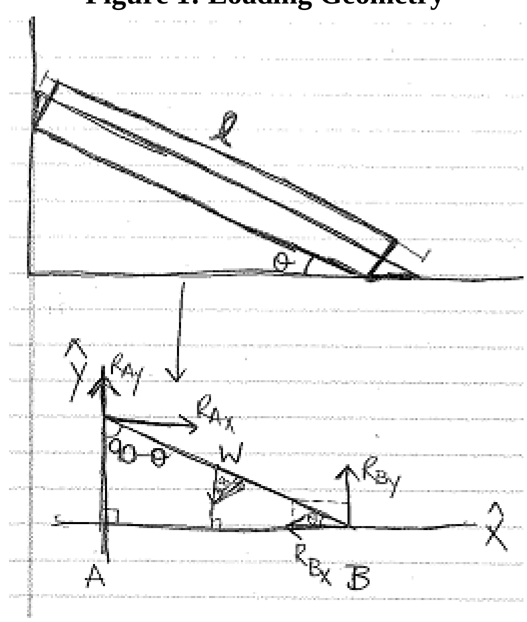
II. Problem Description

Ladder accidents are not uncommon in the US, causing injury to over 164,000 people per year [2]. Even though the safety standards for ladders are specified by OSHA 1910.25 [3], there seem to be few comprehensive analyses of the problem of a ladder leaning against a wall subject to some load such as a person's weight that take into full account the normal and frictional reaction forces.

A ladder standing against a wall can be modeled as a uniform beam in static equilibrium. One approach by Mendelson [4] considers only the axial compression on the ladder in an idealized beam situation; a more advanced treatment considers flexion [6]. However, even the more advanced analysis seems inadequate because it assumes boundary conditions which seem inappropriate.

Here we propose a model of this situation in which a beam is fully treated in flexion and compression using the elastic theory of mechanics of materials. In particular, we consider the following situation:

Figure 1: Loading Geometry



We consider the distributed force of weight W, a point load force on the beam P, and the reactions forces at the two ends of the beam N_A, N_B, F_A, F_B caused by friction and by a normal force in order to determine the solution for the statically-indeterminate problem.

III. Theoretical Background

The theoretical tools used to consider this problem are Newtonian equilibrium, linear axial loading, and Euler deflection.

A. Static Equilibrium

The situation of the beam leaning against the wall necessarily demands analysis in static equilibrium. Static equilibrium [6] is defined as the state where the net external forces and net external moments are zero. Thus, for any object, whether a rigid body or a point particle, we hold the following to be true, for a particle acted on by *i* forces *F* and *j* moments *M* (in as many dimensions as appropriate):

$$\sum_i \mathbf{F}_i = 0$$

$$\sum_i \boldsymbol{M}_i = 0$$

For the beam situation considered here, the external forces and moments are the frictional and normal forces of the floor and wall and the moments caused by these.

B. Axial Loading

In addition to static equilibrium, all physical bodies experience an axial deformation due to the forces applied along the axes of the bodies [7]. This deformation δ is a result of the strain ε proportional to the stress σ engendered by an external force F; assuming the deformation is small and not permanent, we can write linear relationships for basic axial loading as follows:

$$\sigma = \varepsilon E \textit{ where E is Young's Modulus}$$

$$\sigma = \frac{F}{A} \textit{ where A is the cross sectional area normal to the axis}$$

$$\varepsilon = \frac{\delta}{L} \textit{ where L is the length of the body along the axis where the force acts}$$

$$\frac{FL}{EA} = \delta$$

Given multiple forces, we observe that in general, the total deformation along an axis must be the following:

$$\sum_i \frac{\mathbf{L}_i \mathbf{F}_i}{\mathbf{A}_i \mathbf{E}_i} = \boldsymbol{\delta}$$

For the beam problem considered here, all the axial deformation is assumed to be along the long axis of the beam since the short-axis of the beam is very small and would experience negligible deformation.

C. Deflection of Beams

If a beam is acted upon by forces that are transverse to the principal axis of the beam, i.e. along the cross-section, then these transverse forces causing a shear force which result in bending of the beam [7]. Using the linear analysis of deformation with Euler's theory of the beam bending, the deflection of the beam must be the following:

$$\frac{d^4y}{dx^4} = -\frac{1}{EI}w(x)$$

$$\textit{where x is the axial dimension of the beam,}$$

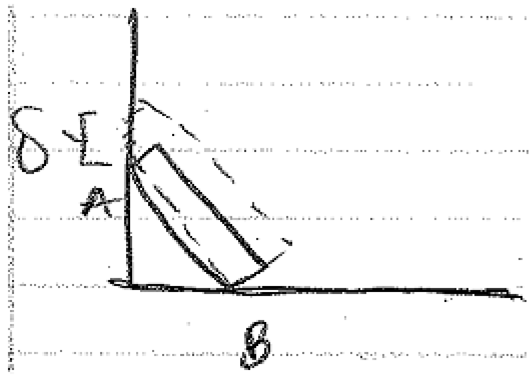
$$w(x) \text{ is the force per unit length along the beam,}$$
$$\text{and } I \text{ is the area moment of inertia of the cross section of the beam}$$

IV. Analysis

Several methods of analysis were attempted for the beam situation and can be found in the Appendix.

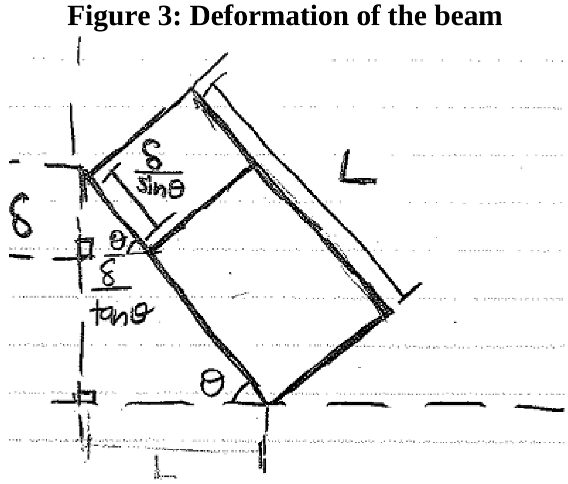
The most successful method follows from direct observation of the deformation of a beam under some force. We noticed that the beam when loaded seems to slide along the wall A by some small amount δ while staying fixed at the floor B (ref: Figure 2).

Figure 2: Sliding along the wall



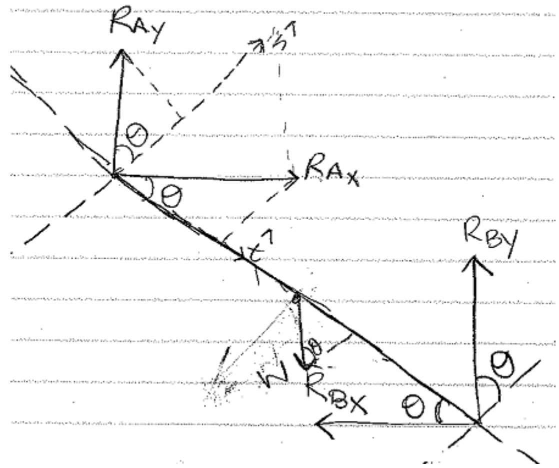
Given the observation of δ and geometry, the axial deformation of the beam must be $\frac{\delta}{\tan(\theta)}$ (Figure 3). However, since the point B remains fixed, it is necessary that the deflection at point A have the following value $y(A) = \delta$ along the transverse axis of the beam. In addition, we require that the length of the beam along its long axis be $L' where\ L' = L - \frac{\delta}{\tan(\theta)}$, conserved in the integral of the deflection $\int_A^{L'} y(x)dx = L$, the original length of the beam (Figure 3).

Figure 3: Deformation of the beam



In order to conduct the analysis of the beam, it is first useful to transform the external reaction forces into components that are along the long axis (tangential) and perpendicular to the long-axis (normal) (Figure 4).

Figure 4: Coordinate Systems



Therefore, considering the normal, frictional, weight, and applied forces on the beam $N_A, N_B, F_A, F_{B,}, P, W$, we can construct the free-body diagrams for the normal and tangential coordinate frames as follow.

Figure 5: Axial Loading (Tangential)

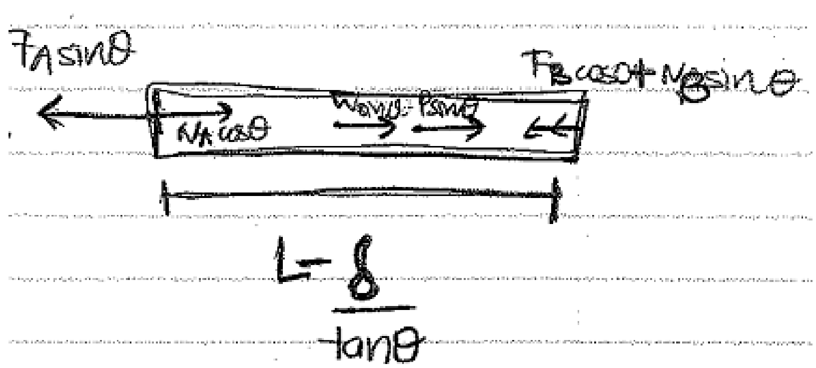
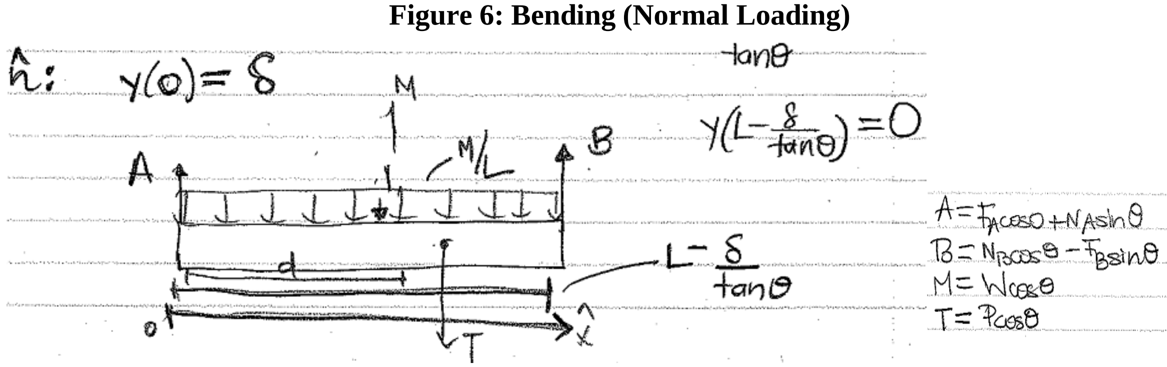


Figure 6: Bending (Normal Loading)



We begin with the equation of static equilibrium. If we consider the forces along the tangential coordinate $\hat{t}$, the normal coordinate $\hat{n}$, and the moment in $\hat{k}$ *where* $\hat{k} = \hat{t} \times \hat{n}$ caused by the forces along the normal coordinate, we can derive the following:

$$\sum_i \mathrm{F}_i\hat{t} = 0 = -Wcos(\theta) - Pcos(\theta) + F_A\cos(\theta) + N_A\sin(\theta) - F_{B,}\sin(\theta) + N_B\cos(\theta)$$

[Equation 1]

$$\sum_i \mathrm{F}_i\hat{t} = 0 = Wsin(\theta) + Psin(\theta) - F_A\sin(\theta) + N_A\cos(\theta) - F_{B,}\cos(\theta) - N_B\sin(\theta)$$

[Equation 2]

$$\sum_i \mathrm{M}_i\hat{k}\ @\ A = 0 = \left(-\frac{L'}{2}\right) * Wcos(\theta) + \left(-\frac{L'}{2} - t\right) * Pcos(\theta) + (L') * (N_B\cos(\theta) - F_{B,}\sin(\theta))$$

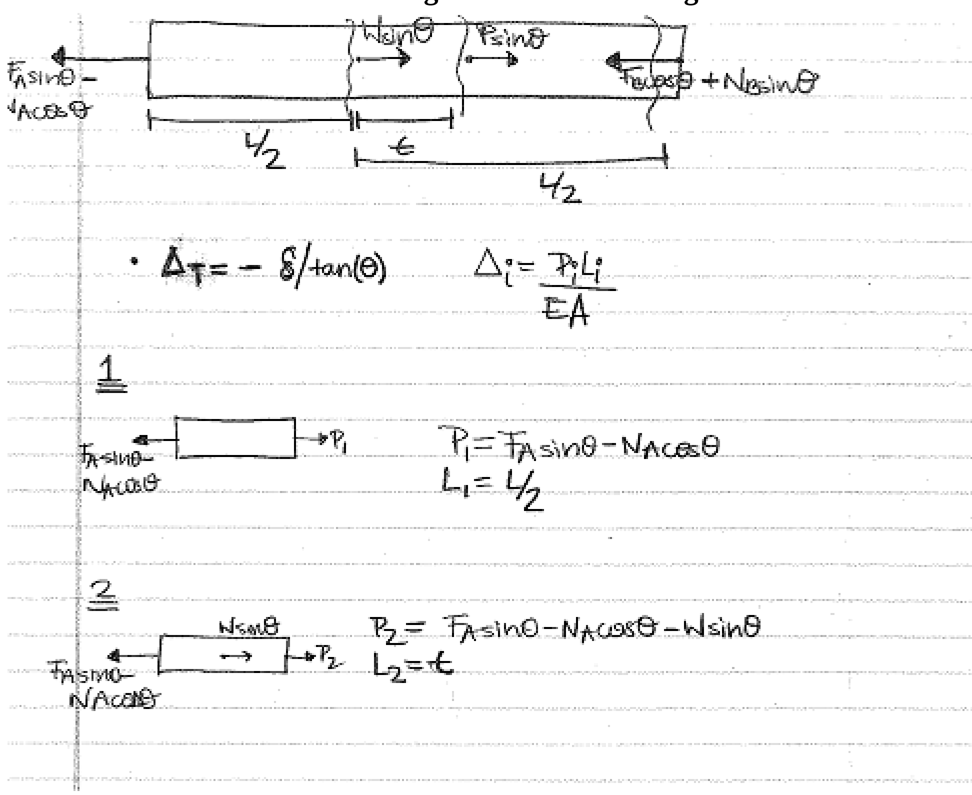
[Equation 3]

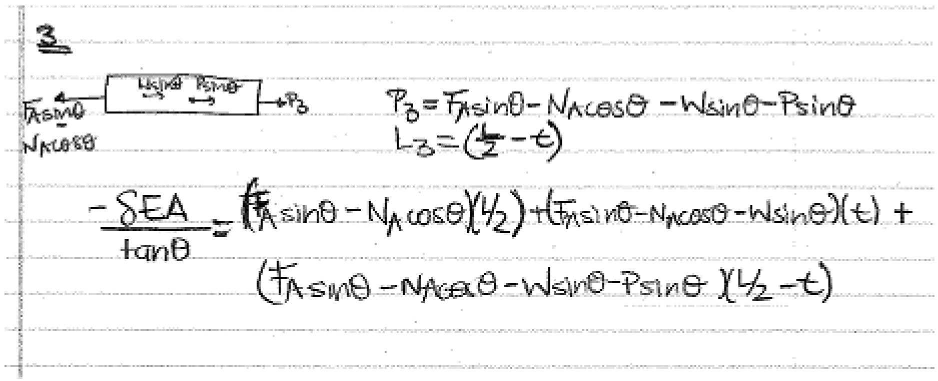
$$where\ L' = L - \frac{\delta}{\tan(\theta)}$$

$$where\ t\ is\ the\ location\ of\ the\ load\ P\ from\ the\ center\ of\ the\ beam$$

Now, we consider the axial loading of the beam by forces in the $\hat{t}$ direction. In order to do so, we take three cuts of the beam in this dimension (**Figure 7**). We note as before that the total deformation in the beam axially must be $-\frac{\delta}{\tan(\theta)}$.

Figure 7: Axial Loading of the Beam





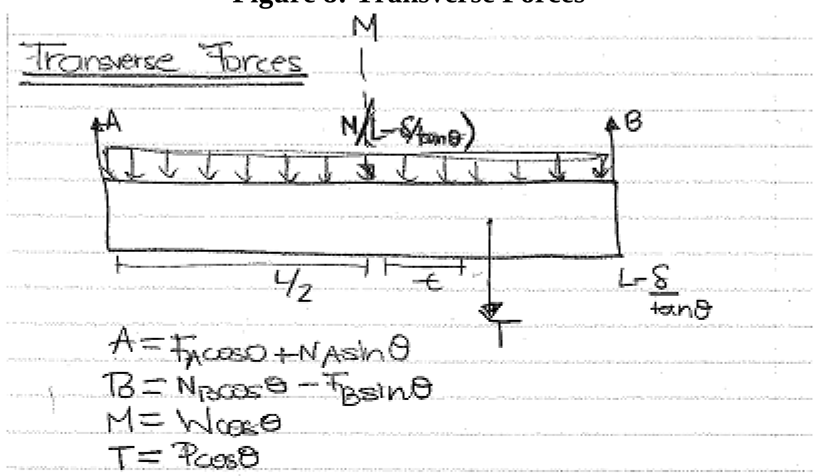
Thus, the axial deflection equation applied yields the following:

$$-\frac{\delta}{\tan(\theta)}EA = L * F_A \sin(\theta) - L * N_A \cos(\theta) - \frac{L}{2} * Wsin(\theta) - \left(\frac{L}{2} - t\right) * Psin(\theta)$$

[Equation 4]

Lastly, we consider the deflection of the beam due to transverse forces in the normal coordinate (Figure 8).

Figure 8: Transverse Forces



Using the deflection equation and discontinuity functions, we obtain the following:

$$\frac{d^3y}{dx^3} = v(x) = \frac{M}{L'}x + A - T\,\langle x - \left(\frac{L'}{2} + t\right)\rangle^0 - B\langle x - L'\rangle^0$$

By applying boundary condition that $y(A) = \delta \; and \; y(B) = 0$ and noticing that there are no applied moments, triple integration yields the following equation:

$$0 = \frac{M}{24}L'^4 + \frac{A}{6}L'^3 + \frac{T}{6}\left(\frac{L'}{2} - t\right)^3 + C_2L' + EI\delta$$

[Equation 5]

However, now, an additional unknown C3 is introduced, giving us 6 unknowns, 5 equations. However, if we require that $\int_A^{L'} y(x)dx = L$, we can also derive one additional equation:

$$0 = \frac{M}{120}L'^4 + \frac{A}{24}L'^4 + \frac{T}{24}\left(\frac{L'}{2} - t\right)^4 + \frac{C_2}{2}L'^2 + EI\delta(L')$$

[Equation 6]

Now, we have 6 unknowns, the four forces, the deformation, and the constant; however, we also have six equations. Thus, we have the following system of equations which provide the solution to this problem:

$$F_A\cos(\theta) + N_A\sin(\theta) - F_{B,}\sin(\theta) + N_B\cos(\theta) + 0 = (W + P) * cos(\theta)$$

$$-F_A\sin(\theta) + N_A\cos(\theta) - F_{B,}\cos(\theta) - N_B\sin(\theta) + 0 = Wsin(\theta) + Psin(\theta)$$

$$0 + 0 + (L') * \left(-F_{B,}\sin(\theta) + N_B\cos(\theta)\right) + 0 = \left(\frac{L'}{2}\right) * Wcos(\theta) + \left(\frac{L'}{2} + t\right) * Pcos(\theta)$$

$$L * F_A\sin(\theta) - L * N_A\cos(\theta) + 0 + 0 + 0 + \frac{\delta}{\tan(\theta)}EA = \frac{L}{2} * Wsin(\theta) + \left(\frac{L}{2} - t\right) * Psin(\theta)$$

$$L * F_A\sin(\theta) - L * N_A\cos(\theta) + 0 + 0 + 0 + \frac{\delta}{\tan(\theta)}EA = \frac{L}{2} * Wsin(\theta) + \left(\frac{L}{2} - t\right) * Psin(\theta)$$

$$+\frac{1}{6}L'^3 * (F_A\cos(\theta) + N_A\sin(\theta)) + 0 + 0 + EI\delta + C_2L' = \frac{-Pcos(\theta)}{6}\left(\frac{L'}{2} - t\right)^3 + \frac{Wcos(\theta)}{24}L'^4$$

$$+\frac{1}{24}L'^4 * (F_A\cos(\theta) + N_A\sin(\theta)) + 0 + 0 + EI\delta L' + \frac{C_2L'^2}{2} = \frac{-Pcos(\theta)}{24}\left(\frac{L'}{2} - t\right)^4 + \frac{Wcos(\theta)}{120}L'^5$$

<u>V. Results</u>

A. Parameters

In order to perform some useful computation on the system, we need parameters L, E, A, P, W, and theta.

From the USPCSC [2], we can acquire some information about the needed parameters of the system:
$\theta = 75\ degrees = 1.30\ rads$ for optimal use of the ladder.

In order to consider a general system, we use a beam with a square unit cross-section and a length 100 times the side of the unit square so as to have a beam which is more than a cube:
$L = 100\ m$ and A= $1\ m^2$ with a moment of inertia $I = \frac{1}{12}\ in^4$.

For P, we choose the 10,000 N, an arbitrary amount close to the weight of ten people. For E, we choose the material fir, which has an E of 13×10^9 Pa and a corresponding density of 470 kg/m^3, yielding a weight W of approximately 470,000 N.

B. Numerical Calculations

There is one issue that makes the numerical calculation and analytical solution of this somewhat impossible. The last two equations use the term L' which actually depend on the deformation δ. As far as I can see, this makes the system un-computable. Thus, more work needs to be done to identify routes to solve this difficult set of equation numerically on the basis of this non-linearity.

In order to approximate a possible solution for now, I will set L' to be some quantity; I will assume that this quantity is actually measurable and will thus be a parameter in our actual solution to this problem. Thus, for the sake of doing a simulation, we can set $L' = 99.995\,m$, which is for a very small δ = 0.005m.

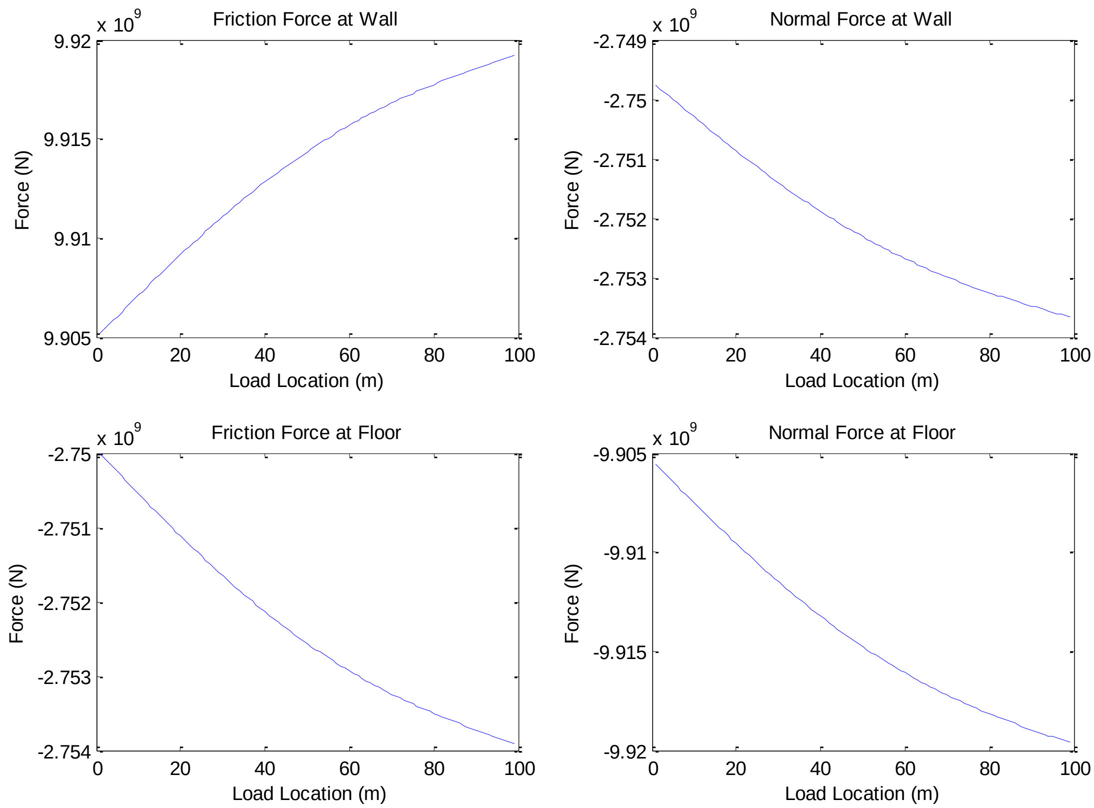
If we choose these parameters, then the solution to the system of equation is the solution to the problem: $\boldsymbol{A} * \boldsymbol{F} = \boldsymbol{C}$ where A is the matrix of coefficients, F is the vector of unknowns, and C is the vector of knows.

$\cos(\theta)$	$\sin(\theta)$	$-\sin(\theta)$	$\cos(\theta)$	0	0	F_A		$(W+P)*cos(\theta)$
$-\sin(\theta)$	$\cos(\theta)$	$-\cos(\theta)$	$-\sin(\theta)$	0	0	N_A		$Wsin(\theta) + Psin(\theta)$
0	0	$(-L'\sin(\theta))$	$(L'\cos(\theta))$	0	0	$F_{B,}$	=	$\left(\frac{L'}{2}\right) * Wcos(\theta) + \left(\frac{L'}{2} + t\right) * Pcos(\theta)$
$Lsin(\theta)$	$-Lcos(\theta)$	0	0	$\frac{1}{\tan(\theta)} EA$	0	N_B		$\frac{L}{2} * Wsin(\theta) + \left(\frac{L}{2} - t\right) * Psin(\theta)$
$\frac{\cos(\theta)}{6} L'^3$	$\frac{\sin(\theta)}{6} L'^3$	0	0	EI	L'	δ		$\frac{-Pcos(\theta)}{6}\left(\frac{L'}{2} - t\right)^3 + \frac{Wcos(\theta)}{24} L'^4$
$\frac{\cos(\theta)}{24} L'^4$	$\frac{\sin(\theta)}{24} L'^4$	0	0	EIL'	$\frac{L'^2}{2}$	C_2		$\frac{-Pcos(\theta)}{24}\left(\frac{L'}{2} - t\right)^4 + \frac{Wcos(\theta)}{120} L'^5$

[Equation 7]

Thus, for this linear and somewhat troubling approximation, we can use a numerical solution to determine the unknowns. I compute here the reaction forces as a function of *d*, the distance of the load force *P* from the end of the beam at A, considering $x = 0$ to be at A (Figure 10) . The code is attached to allow computation for different kinds of conditions.

Figure 10: Reaction Force Distribution for $L' = 99.995\ m$



VI. Discussion

A. Interpretation

From looking at the system of equations which describes the system (equations 1-6), we note than it is non-linear, have products of the variables of the systems in it, so as to not be factorizable into a matrix form for solution. Unsure in how to deal with this at this time, an approximation was used, in setting one of the variables as a parameter in order to achieve a factorizable matrix (equation 7).

However, when this approach was taken, it was found that this analysis yielded off results. Some of the reaction forces were negative in this regime, which is impossible because the normal forces and the frictional motions can only act to oppose the given loads, and never cause some kind of load themselves. Thus, the solution does not make physical sense.

The non-physicality points to one explanation. The approximation that uncouples one of the parameters cannot work as the system must be strongly non-linear and thus require a novel numerical method to solve that can deal with high order polynomial non-linearity in the non-factorizable form of the equations.

An alternative explanation is that some algebraic exists in the equations, but in the absence of noticing this, it is hard to determine whether there are issues with the analysis.

B. Future Work

First of all, the analysis needs to be checked and verified for errors. Second, some kind of different numerical approach needs to be taken to solve these equations in their legitimate form. Third, the model needs to be compared to experimental data to verify its usefulness.

VII. Appendix

- Appendix 1: contains the work from the semester's worth of analysis
- Appendix 2: I attach below the relevant MatLab code.

VIII. References

1. Artistic Ladder Picture http://www.dreamstime.com/stock-images-wooden-ladder-against-wall-image1082754
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