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John P. Georges

Trinity College, john.georges@trincoll.edu

David W. Mauro

Trinity College, david.mauro@trincoll.edu

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ON REGULAR GRAPHS OPTIMALLY LABELED WITH A CONDITION AT DISTANCE TWO*

JOHN P. GEORGES[†] AND DAVID W. MAURO[†]

Abstract. For positive integers $j \geq k$, the $\lambda_{j,k}$ -number of graph G is the smallest span among all integer labelings of $V(G)$ such that vertices at distance two receive labels which differ by at least k and adjacent vertices receive labels which differ by at least j . We prove that the $\lambda_{j,k}$ -number of any r -regular graph is no less than the $\lambda_{j,k}$ -number of the infinite r -regular tree $T_\infty(r)$. Defining an r -regular graph G to be (j, k, r) -optimal if and only if $\lambda_{j,k}(G) = \lambda_{j,k}(T_\infty(r))$, we establish the equivalence between (j, k, r) -optimal graphs and r -regular bipartite graphs with a certain edge coloring property for the case $\frac{j}{k} > r$. The structure of r -regular optimal graphs for $\frac{j}{k} \leq r$ is investigated, with special attention to $\frac{j}{k} = 1, 2$. For the latter, we establish that a $(2, 1, r)$ -optimal graph, through a series of edge transformations, has a canonical form. Finally, we apply our results on optimality to the derivation of the $\lambda_{j,k}$ -numbers of prisms.

Key words. $L(j, k)$ -labeling, regular graph, prism

AMS subject classification. 05C

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1. Introduction. For positive integers j and k with $j \geq k$, an $L(j, k)$ -labeling of graph G is an assignment L of nonnegative integers to the vertices of G such that

- (1) $|L(v) - L(u)| \geq j$ if v and u are adjacent, and
- (2) $|L(v) - L(u)| \geq k$ if v and u are distance two apart.

Elements of the image of L are called *labels*, and the *span* of L , denoted $s(L)$, is the difference between the largest and smallest labels. The minimum span taken over all $L(j, k)$ -labelings of G , denoted $\lambda_{j,k}(G)$, is called the $\lambda_{j,k}$ -number of G , and if L is a labeling with minimum span, then L is called a $\lambda_{j,k}$ -labeling of G . Unless otherwise stated, we shall assume with no loss of generality that the minimum label of $L(j, k)$ -labelings of G is 0.

A variation of Hale's channel assignment problem [12], the problem of labeling a graph with a condition at distance two, was first investigated in the case $j = 2$ and $k = 1$ by Griggs and Yeh [11]. Other authors have since explored the $\lambda_{2,1}$ -numbers of graphs in various classes, as well as relationships between $\lambda_{2,1}(G)$ and other invariants of G (see [2, 6, 9, 10, 13, 14, 16, 17, 18, 19]). Additionally, properties of $\lambda_{j,k}$ -numbers have been investigated in [1, 4, 5] and [7].

In this paper, we develop the notion of optimality among r -regular graphs by considering the $\lambda_{j,k}$ -number of the infinite r -regular tree $T_\infty(r)$, $r \geq 2$ [4]. We show in section 2 that $\lambda_{j,k}(G) \geq \lambda_{j,k}(T_\infty(r))$ for any r -regular graph G , and we define G to be (j, k, r) -optimal if and only if the equality holds. In section 3, we consider the structure of (j, k, r) -optimal graphs for $\frac{j}{k} > r$ and show that (j, k, r) -optimal graphs are bipartite with block edge coloring number r . In section 4, we define the notion of cyclic optimality in the exploration of the case $\frac{j}{k} \leq r$, with special attention to $j = k = 1$. We consider the structure of $(2, 1, r)$ -optimal graphs in section 5 and

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[†]Department of Mathematics, Trinity College, Hartford, CT 06106 (john.georges@trincoll.edu, David.Mauro@trincoll.edu).

establish a canonical form for such graphs. Finally, in section 6, we use the results in the preceding sections to determine the $\lambda_{2,1}$ -numbers and $\lambda_{1,1}$ -numbers of prisms.

2. Definitions and preliminary results. Throughout the paper, $x \equiv y \pmod n$ shall mean that $x - y$ is divisible by n , and $x = y \pmod n$ shall mean that x is set equal to the remainder that results when y is divided by n .

Let G be a graph and let L be an $L(j, k)$ -labeling of G . Then $M_i(G, L) = \{v \in V(G) \mid L(v) = i\}$ and $m_i(G, L) = |M_i(G, L)|$. When there is no possibility of confusion, reference to G and L will be suppressed.

Georges and Mauro [4] derived $\lambda_{j,k}(T_\infty(r))$ for all j, k , and r , including the two particular cases which will be of importance to this paper.

THEOREM 2.1. For $\frac{j}{k} \geq r$, $\lambda_{j,k}(T_\infty(r)) = j + (2r - 2)k$.

THEOREM 2.2.

$$\lambda_{j,1}(T_\infty(r)) = \begin{cases} r + 2j - 2 & \text{if } j \leq r, \\ j + 2r - 2 & \text{if } j \geq r. \end{cases}$$

We next show that $\lambda_{j,k}(T_\infty(r))$ is a lower bound for the $\lambda_{j,k}$ -numbers of all r -regular graphs, which in turn will serve to motivate the notion of (j, k, r) -optimality.

THEOREM 2.3. If G is a connected r -regular graph, then $\lambda_{j,k}(G) \geq \lambda_{j,k}(T_\infty(r))$.

Proof. Suppose L is an $L(j, k)$ -labeling of G with span $s(L)$. It suffices to show that L induces an $L(j, k)$ -labeling of $T_\infty(r)$ with span $s(L)$.

Let v_{n_0} be an arbitrarily selected vertex in $V(G)$ and let the neighbors of v_{n_0} be $v_{n_1}, v_{n_2}, \dots, v_{n_r}$. We assign the label $L(v_{n_0})$ to the root w_0 of $T_\infty(r)$, and we assign the labels $L(v_{n_1}), L(v_{n_2}), \dots, L(v_{n_r})$ to the children w_1, w_2, \dots, w_r of w_0 , respectively. The $r - 1$ children of w_i may then be assigned the labels of the neighbors of v_{n_i} which have not already been assigned to the parent of w_i . The result follows by induction. \square

For the case $(j, k) = (2, 1)$ and $r \geq 2$, the well-known inequality $\lambda_{2,1}(G) \geq r + 2$ was used by Jha [13] in his consideration of the λ -number of the Kronecker product of cycles. There, he called those products with $\lambda_{2,1}$ -numbers equal to the lower bound *optimal*. We extend his terminology to the consideration of optimal (j, k, r) -labelings of r -regular graphs as follows.

DEFINITION 2.4. For $r \geq 2$, the graph G is said to be (j, k, r) -optimal if and only if G is r -regular and $\lambda_{j,k}(G) = \lambda_{j,k}(T_\infty(r))$. If L is a $\lambda_{j,k}$ -labeling of a (j, k, r) -optimal graph G , then L is said to be a (j, k, r) -optimal labeling of G . We denote the set of (j, k, r) -optimal graphs by $\Gamma(j, k, r)$.

It follows from Theorems 2.1 and 2.2 that G is $(j, 1, r)$ -optimal if and only if $\lambda_{j,1}(G) = \lambda_{j,1}(T_\infty(r))$.

3. Optimality with $\frac{j}{k} > r$. In this section we consider the structure of (j, k, r) -optimal graphs for $\frac{j}{k} > r$. As noted in Theorem 2.1, such graphs have $\lambda_{j,k}$ -number $j + (2r - 2)k$.

THEOREM 3.1. For $\frac{j}{k} > r$, if G is (j, k, r) -optimal, then G is bipartite with $|V(G)| \equiv 0 \pmod{2r}$.

Proof. Let L be a (j, k, r) -optimal labeling of G . Since the span of L is $j + (2r - 2)k$, each vertex in $V(G)$ has a label in exactly one of the three intervals $X_1 = [0, (r - 1)k]$, $X_2 = [(r - 1)k + 1, j + (r - 1)k - 1]$, and $X_3 = [j + (r - 1)k, j + (2r - 2)k]$. Suppose $L(v) \in X_2$, and suppose that exactly m neighbors of v have labels less than $L(v)$, $0 < m < r$. Then the smallest label among the neighbors of v is at most $L(v) - j - (m - 1)k$, and the largest label among the neighbors of v is at least $L(v) + j + (r - m - 1)k$. The span of L is

thus at least $L(v) + j + (r - m - 1)k - (L(v) - j - (m - 1)k) = 2j + (r - 2)k > j + (2r - 2)k$, a contradiction. Arguing similarly, if $m = 0$, then the largest label among the neighbors of v is at least $j + (2r - 2)k + 1$, a contradiction. And if $m = r$, then the smallest label among the neighbors of v is at most -1 , another contradiction. Hence each label assigned by L is in X_1 or X_3 . For $i \in \{1, 3\}$, no two distinct vertices in X_i are adjacent since the length X_i is less than j . Hence, G is bipartite.

Now let $v \in V(G)$ with $L(v) \in X_1$. Then the r neighbors of v have labels in X_3 . Since the neighbors of v are pairwise distance two apart, their labels pairwise differ by at least k , and hence the labels of the neighbors of v must be $j + (r - 1 + i)k$, $0 \leq i \leq r - 1$. A similar argument demonstrates that each vertex with label in X_3 has neighbors with labels ik , $0 \leq i \leq r - 1$. Thus, there are exactly $2r$ distinct labels under L with non-zero multiplicity; in X_1 , these are $0, k, 2k, \dots, (r - 1)k$, and in X_3 these are $j + (r - 1)k, j + rk, \dots, j + (2r - 2)k$.

Let x_1 and x_3 be labels assigned by L in X_1 and X_3 , respectively. Then we have seen that each vertex in M_{x_1} is adjacent to some vertex in M_{x_3} . Moreover, due to the distance two condition, no two vertices in M_{x_1} can be adjacent to the same vertex in M_{x_3} . Thus $m_{x_1} \leq m_{x_3}$. Similarly, $m_{x_1} \geq m_{x_3}$, implying $m_{x_1} = m_{x_3}$. Hence, since L partitions $V(G)$ into $2r$ nonempty labeling classes, $|V(G)| = 2rm_{x_1}$, from which the result follows. \square

We next characterize those graphs in $\Gamma(j, k, r)$, $\frac{j}{k} > r$. It can be easily seen that $K_{r,r}$, the complete r -regular bipartite graph of smallest order, is the graph of smallest order in $\Gamma(j, k, r)$ (see [7]). We also point out that the converse of Theorem 3.1 is not true. For example, the graph $3Q_3$, the sum of 3 copies of the 3-cube, is a 3-regular bipartite graph with order 24; however, for $\frac{j}{k} > 3$, $\lambda_{j,k}(3Q_3) = \lambda_{j,k}(Q_3) = j + 5k$ (see [5]). Alternatively, we observe that a graph G is (j, k, r) -optimal if and only if each component of G is (j, k, r) -optimal. So, since Theorem 3.1 implies that Q_3 is not (j, k, r) -optimal, neither is $3Q_3$.

THEOREM 3.2. *Let G be an r -regular graph with $|V(G)| \equiv 0 \pmod{2r}$. Then $G \in \Gamma(j, k, r)$ if and only if there exists a partition of $V(G)$ into sets $A_0, A_1, A_2, \dots, A_{r-1}, B_0, B_1, B_2, \dots, B_{r-1}$ such that for each i , $0 \leq i \leq r - 1$, every vertex v in A_i (resp., B_i) has exactly one neighbor in B_j (resp., A_j), $0 \leq j \leq r - 1$.*

Proof. (\Rightarrow) Let L be a (j, k, r) -optimal labeling of G . Then the result follows from the proof of Theorem 3.1 with A_i equal to the set of vertices with label ik under L and B_i equal to the set of vertices with label $j + (r - 1 + i)k$ under L , $0 \leq i \leq r - 1$.

(\Leftarrow) The vertices in each set A_i (resp., B_i) are pairwise distance three or more apart. Additionally, for $i \neq j$, each vertex in A_i (resp., B_i) is distance two or more from each vertex in A_j (resp., B_j). Thus, we form an $L(j, k)$ -labeling L of G by assigning ik to each vertex in A_i , $0 \leq i \leq r - 1$, and $j + (r - 1 + i)k$ to each vertex in B_i , $0 \leq i \leq r - 1$. Since the span of L is $j + (2r - 2)k$, we are done. \square

DEFINITION 3.3. *Let $B = X \cup Y$ be an r -regular bipartite graph and let \mathcal{L} be an edge coloring of B such that*

- (i) *for each $x \in X$, the edges incident to x are assigned the same color under \mathcal{L} ,*
- (ii) *for each $y \in Y$, the edges incident to y are assigned distinct colors under \mathcal{L} .*

Then \mathcal{L} is called an X -block coloring of B . We denote the minimum number of colors assigned by X -block colorings of B by $\zeta_X(B)$, and if \mathcal{L} is an X -block coloring of B which assigns exactly $\zeta_X(B)$ distinct colors, then \mathcal{L} is called a minimum X -block coloring of B .

We observe that $r \leq \zeta_X(B) \leq |X|$. To illustrate, we note that for either bipartition $X \cup Y$ of $K_{r,r}$, $\zeta_X(K_{r,r}) = r$ and for any bipartition $X \cup Y$ of C_6 , $\zeta_X(C_6) = 3$.

THEOREM 3.4. *Let G be an r -regular bipartite graph with bipartition W_1 and W_2 . Then for $\frac{j}{k} > r$, $G \in \Gamma(j, k, r)$ if and only if $\zeta_{W_1}(G) = \zeta_{W_2}(G) = r$.*

Proof. (\Rightarrow) By Theorem 3.2, we let $W_1 = \bigcup_{i=1}^{r-1} A_i$ and $W_2 = \bigcup_{i=1}^{r-1} B_i$. We form a W_1 (resp., W_2)-block coloring using r colors c_0, c_1, \dots, c_{r-1} by assigning color c_i to each edge which is adjacent to some vertex in A_i (resp., B_i), $0 \leq i \leq r-1$. But $\zeta_{W_1}(G) \geq r$ (resp., $\zeta_{W_2}(G) \geq r$) since the degree of each vertex in B_0 (resp., A_0) is r . So $\zeta_{W_1}(G) = r$ (resp., $\zeta_{W_2}(G) = r$).

(\Leftarrow) For $i = 1, 2$, let C_i be minimum W_i -block colorings of G . We produce a vertex labeling L as follows: for each vertex v in W_1 whose incident edges receive color c_i under C_1 , $0 \leq i \leq r-1$, let $L(v) = ik$, and for each vertex in W_2 whose incident edges receive color c_i under C_2 , let $L(v) = j + (r-1)k + ik$. To see that L is a (j, k) -labeling, we note that the difference between the largest label among the vertices in W_1 and the smallest label among the vertices in W_2 is $j + (r-1)k - (r-1)k = j$, implying that the labels of adjacent vertices differ by at least j . To show that the distance two condition is satisfied by L , it suffices to show that two vertices distance two apart receive different labels under L . If x_1 and x_2 are distance two apart with $L(x_1) = L(x_2)$ and $x_1, x_2 \in W_1$ (resp., W_2), then there exists vertex $y \in W_2$ (resp., W_1) and edges $\{x_1, y\}$ and $\{x_2, y\}$ which receive the same color under C_1 (resp., C_2), a contradiction. \square

We illustrate a 3-regular bipartite graph $B = X \cup Y$ on 12 vertices with $\zeta_X(B) = 3$ and $\zeta_Y(B) = 5$. For $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, let the neighborhood set of x_i , denoted $N(x_i)$, be as follows:

$$\begin{aligned} N(x_1) &= \{y_1, y_2, y_3\}, \\ N(x_2) &= \{y_4, y_5, y_6\}, \\ N(x_3) &= \{y_1, y_3, y_5\}, \\ N(x_4) &= \{y_2, y_4, y_6\}, \\ N(x_5) &= \{y_1, y_3, y_4\}, \\ N(x_6) &= \{y_2, y_5, y_6\}. \end{aligned}$$

Then for $1 \leq i \leq 3$, assigning color i to the edges incident to x_{2i-1} and x_{2i} shows that $\zeta_X(B) = 3$. On the other hand, examination of the neighborhood sets of each y_i gives $\zeta_Y(B) = 5$.

THEOREM 3.5. *For $r \geq 3$, let j, k, j' , and k' be integers such that $\frac{j}{k} > r$ and $\frac{j'}{k'} > r$. Then $\Gamma(j, k, r) = \Gamma(j', k', r)$.*

Proof. Let $G \in \Gamma(j, k, r)$. By Theorem 3.1, G is bipartite, so G can be expressed $X \cup Y$. This implies $\zeta_X(G) = \zeta_Y(G) = r$ by Theorem 3.4, which in turn implies (also by Theorem 3.4) that $G \in \Gamma(j', k', r)$. Thus $\Gamma(j, k, r) \subseteq \Gamma(j', k', r)$. A similar argument shows $\Gamma(j', k', r) \subseteq \Gamma(j, k, r)$. \square

Let $x = \frac{j}{k}$. In [4], it is shown that for any graph G , the function $\lambda_x(G) = \frac{1}{k} \lambda_{j,k}(G)$ is continuous in x on the set of rationals greater than or equal to 1. (Here, continuity at rational number $x \geq 1$ means for any real $\epsilon > 0$, there exists real $\delta > 0$ such that for rational $q \geq 1$ within δ of x , $\lambda_x(G)$ is within ϵ of $\lambda_q(G)$.) Additionally, we have seen that, if $H \in \Gamma(j, k, r)$, $\frac{j}{k} > r$, then $\lambda_x(H) = x + (2r - 2)$. Thus $\lambda_r(H) = r + (2r - 2)$, which establishes that $H \in \Gamma(ar, a, r)$ for $a \in \mathbb{Z}^+$. It follows that for $\frac{j}{k} > r$, $\Gamma(j, k, r) \subseteq \Gamma(ar, a, r)$. We point out, however, that for $\frac{j}{k} > r$, $\Gamma(j, k, r)$ and $\Gamma(ar, a, r)$ are not equal. As an example, K_3 , which is not bipartite and hence not optimal for $\frac{j}{k} > 2$, is a member of $\Gamma(2, 1, 2)$.

4. Optimality with $\frac{j}{k} \in \mathbb{Z}^+$ and $\frac{j}{k} < r$. In this section, we investigate the structure of $\Gamma(j, k, r)$ for $\frac{j}{k}$ an integer. Since $\lambda_{j,k}(G) = k\lambda_{c,1}(G)$ for $\frac{j}{k} = c \in \mathbb{Z}^+$, it will suffice to assume $k = 1$.

We begin with a consideration of $\Gamma(r - 1, 1, r)$ for $r \geq 2$.

THEOREM 4.1. *For $r \geq 2$, $\Gamma(r - 1, 1, r) \subseteq \Gamma(r, 1, r)$.*

Proof. If $G \in \Gamma(r - 1, 1, r)$, then $\lambda_{r-1,1}(G) = 3r - 4$ by Theorem 2.2. Let L be a $\lambda_{r-1,1}$ -labeling of G . Then $L'(x) = L(x) + \lfloor \frac{L(x)}{r-1} \rfloor$ is an $L(r, 1)$ -labeling of G with span $3r - 2$. \square

It follows from the discussion at the end of section 3 that for all $a \in \mathbb{Z}^+$ and $\frac{j'}{k'} > r$, $\Gamma(a(r - 1), a, r) \cup \Gamma(j', k', r) \subseteq \Gamma(ar, a, r)$. We next turn our attention to a special class of optimal labelings.

DEFINITION 4.2. *Let $1 \leq j \leq r$ and let L be a $(j, 1, r)$ -optimal labeling of r -regular graph G . Then L is said to be a $(j, 1, r)$ -cyclically optimal labeling of G if and only if for any adjacent vertices v_i and $v_{i'}$ in $V(G)$, $L(v_i) \notin \{L(v_{i'}) \pm j' \pmod{\lambda_{j,1}(G) + 1} \mid 0 \leq j' \leq j - 1\}$. If G has a $(j, 1, r)$ -cyclically optimal labeling, then G is said to be $(j, 1, r)$ -cyclically optimal; otherwise, G is $(j, 1, r)$ -acyclically optimal. We denote the collection of $(j, 1, r)$ -cyclically optimal graphs by $\Gamma_c(j, 1, r)$.*

To illustrate, we give a $(3, 1, 4)$ -cyclically optimal labeling of a graph in Figure 4.1. We also point out that K_3 is an element of $\Gamma(2, 1, 2)$ but not of $\Gamma_c(2, 1, 2)$.

We also note that $\Gamma_c(1, 1, r)$ necessarily equals $\Gamma(1, 1, r)$.

THEOREM 4.3. *Let G be a $(j, 1, r)$ -cyclically optimal graph, where $j \leq r$. Then $|V(G)| \equiv 0 \pmod{r + 2j - 1}$.*

Proof. Let L be a $(j, 1, r)$ -cyclically optimal labeling of graph G with span $2j + r - 2$ by Theorem 2.2. It suffices to show $m_0 = m_1 = \dots = m_{r+2j-2}$.

By the definition of cyclic labeling, the r neighbors of any vertex v with label $L(v) = x$ must have labels which are precisely the elements of $S_x = \{(L(v) + j + i) \pmod{r + 2j - 1} \mid 0 \leq i \leq r - 1\}$. Thus, since v cannot be adjacent to two vertices with the same label, we have $m_x \leq m_y$ for every y in S_x . But if $y \in S_x$, then $x \in S_y$, so $m_y \leq m_x$. Thus $m_i = m_{j+i} = m_{j+i+1} = m_{i+1}$ for $0 \leq i \leq j + r - 3$, giving the result. \square

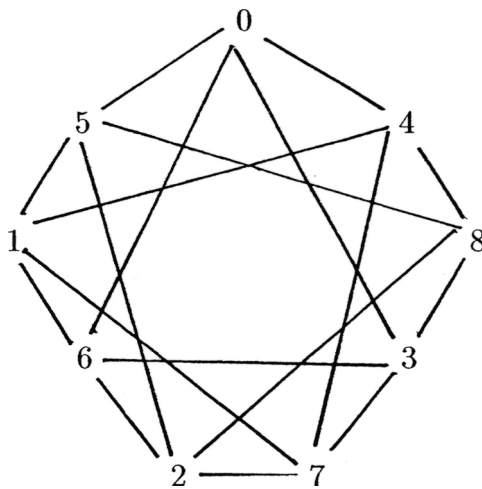


FIG. 4.1. A $(3, 1, 4)$ -cyclically optimal labeling of graph G .

COROLLARY 4.4. *If $G \in \Gamma(1, 1, r)$, then $|V(G)| \equiv 0 \pmod{r + 1}$.*

Proof. If $G \in \Gamma(1, 1, r)$, then G is necessarily $(1, 1, r)$ -cyclically optimal. The result follows immediately from Theorem 3.1. \square

We note that the converse to Corollary 4.4 is not true since the $\lambda_{1,1}$ -number of $C_4 + C_5$ (the sum of C_4 and C_5) is 4.

THEOREM 4.5. *Let $r \geq 2$. If for fixed j , $1 \leq j \leq r$, G is bipartite and $(j, 1, r)$ -cyclically optimal, then $|V(G)| \equiv 0 \pmod{2r + 4j - 2}$.*

Proof. Let L be a $(j, 1, r)$ -cyclically optimal labeling of the bipartite graph $G = X \cup Y$. As in the proof of Theorem 3.1, it can be easily shown that each of the $r + 2j - 1$ labels has the same multiplicity. Since L is cyclic, the subgraph of G induced by $M_i \cup M_{i+j} \cup M_{i+2j}$, $0 \leq i \leq r - 2$, is 2-regular and thus is a sum of even cycles each of which has order divisible by 6. It follows that $|M_i \cap X| = |M_i \cap Y|$. Hence, each m_i is even, which establishes the theorem. \square

We now give a constructive characterization of $\Gamma_c(j, 1, r)$. Let n and h be fixed, $n \geq 3$ and $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$. Then the generalized h -cycle on n vertices, denoted ${}_h C_n$, is the graph with vertex set $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ and edge set $\{v_i v_s \mid 0 \leq i \leq n - 1$ and $s = (i + l) \pmod{n}, 1 \leq l \leq h\}$. We note that ${}_h C_n$ is isomorphic to C_n and K_n when, respectively, $h = 1$ and $h = \lfloor \frac{n}{2} \rfloor$.

Now fix r, j , and $m, j \leq r$. For $1 \leq i \leq m$, let G_i be the graph on $r + 2j - 1$ vertices $v_{i,0}, v_{i,1}, v_{i,2}, \dots, v_{i,r+2j-2}$ such that for all $l, v_{i,l}$ is adjacent to precisely every vertex in $V(G_i)$ except $v_{i,l \pm x \pmod{r+2j-1}}, 0 \leq x \leq j - 1$. (We note that G_i is isomorphic to ${}_h C_n$, where $h = j - 1$ and $n = r + 2j - 1$.) Then it is easily verified that G_i is in $\Gamma_c(j, 1, r)$ and that the labeling L_i of G_i such that $L_i(v_{i,x}) = x$ is a $(j, 1, r)$ -cyclically optimal labeling. Consequently, the graph $G = \sum_{i=1}^m G_i$ is a $(j, 1, r)$ -cyclically optimal graph and the labeling of G given by $L(v_{i,x}) = x$ is a $(j, 1, r)$ -cyclically optimal labeling.

Let \mathcal{M}_0 be the singleton set containing G , and let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \dots$ be defined recursively as follows: for $y \geq 1, G'' \in \mathcal{M}_y$ if and only if for some graph $G' \in \mathcal{M}_{y-1}$ with edges $v_{i_1, x_1} v_{i_1, x_2}$ and $v_{i_3, x_1} v_{i_2, x_2}$, G'' results from the following edge transpositions on G' :

1. Delete $v_{i_1, x_1} v_{i_1, x_2}$.
2. Delete $v_{i_3, x_1} v_{i_2, x_2}$.
3. Add $v_{i_1, x_1} v_{i_2, x_2}$.
4. Add $v_{i_1, x_2} v_{i_3, x_1}$.

Then by induction, each graph G in $\bigcup_{y=0} \mathcal{M}_y$ is r -regular with cyclic labeling L , since the effect of the edge transpositions is to redirect two edges from vertices labeled x_1 and x_2 to vertices with labels x_1 and x_2 . Thus $\bigcup_{y=0} \mathcal{M}_y \subseteq \Gamma_c(j, 1, r)$.

To show that $\Gamma_c(j, 1, r) \subseteq \bigcup_{y=0} \mathcal{M}_y$, let G be a $(j, 1, r)$ -cyclically optimal graph and let L be a $(j, 1, r)$ -cyclically optimal labeling of G . Then Theorem 4.3 implies that $|V(G)| = c(r + 2j - 1)$ for some c and that we may thus denote the vertices of G by $v_{i,z}, 1 \leq i \leq c$ and $0 \leq z \leq r + 2j - 2$, where $L(v_{i,z}) = z$. Furthermore, since the r neighbors of $v_{i,x}$ necessarily have labels $(x + y) \pmod{r + 2j - 1}, j \leq y \leq r + j - 1$, it is the case that for every i_1, i_2, x_1, x_2 such that $i_1 \neq i_2$ and v_{i_1, x_1} is adjacent to v_{i_2, x_2} , there exists $i_3 \neq i_1$ such that v_{i_1, x_2} is adjacent to v_{i_3, x_1} . Hence, for $i_1 \neq i_2$, an r -regular graph G' may be formed by executing the following algorithm, which may be thought of as a reversal of the edge manipulation algorithm given above:

- a. Delete $v_{i_1, x_1} v_{i_2, x_2}$.
- b. Delete $v_{i_1, x_2} v_{i_3, x_1}$.
- c. Add $v_{i_1, x_1} v_{i_1, x_2}$.
- d. Add $v_{i_3, x_1} v_{i_2, x_2}$.

Moreover, the vertex labeling $L'(v_{i,x})$ of G' given by $L' = L$ is a $(j, 1, r)$ -optimal labeling, since the effect of these edge manipulations is to redirect two edges from vertices labeled x_1 and x_2 to vertices labeled x_1 and x_2 .

As compared to G , the graph G' produced by this algorithm has 1 (or 2) fewer edges of the form $v_{a,x_i}v_{b,x_h}$ where $a \neq b$, and 1 (or 2) more edges of the form $v_{c,x_i}v_{c,x_h}$. The algorithm may thus be iterated sufficiently many times to produce $\sum_{i=1}^m G_i$, each of whose edges is of the form $v_{c,x_i}v_{c,x_h}$. Hence, $\Gamma_c(j, 1, r) \subseteq \bigcup_{y=0}^m \mathcal{M}_y$, which in turn implies the following.

THEOREM 4.6. *Every $(j, 1, r)$ -cyclically optimal graph yields, through a sequence of edge transpositions, a graph isomorphic to a sum of copies of ${}_{j-1}C_{r+2j-1}$.*

From this construction, we have the following.

COROLLARY 4.7. *A connected graph G is $(j, 1, r)$ -cyclically optimal if and only if there exists a partition $\{V_0, V_1, V_2, \dots, V_{r+2j-2}\}$ of $V(G)$ such that, for $0 \leq i \leq r + 2j - 2$, each vertex in V_i is adjacent to exactly one vertex in $V_{(i+i') \pmod{r+2j-1}}$, $j \leq i' \leq j + r - 1$. Necessarily, the sets in the partition are of equal size.*

5. Optimality with $\frac{j}{k} = 2$. In this section, we investigate the graphs in $\Gamma(2, 1, r)$, $r \geq 2$, each of which has $\lambda_{2,1}$ -number equal to $r + 2$. Since, in general, not all $(2, 1, r)$ -optimal labelings of r -regular graphs are cyclic, then the $r + 3$ labels given by an optimal labeling L need not have equal multiplicities. However, as we shall see, the multiplicities of labels under a $(2, 1, r)$ -optimal labeling L do possess certain regularities.

We note that for $r = 2$, any graph in $\Gamma(2, 1, r)$ is a cycle C_n . Moreover, since $\lambda_{2,1}(C_n) = 4 = r + 2$, it follows that $\Gamma(2, 1, 2) = \{C_n | n \geq 3\}$. It thus suffices to consider $r \geq 3$.

THEOREM 5.1. *Let G be a $(2, 1, r)$ -optimal graph, where $r \geq 3$, and let L be a $\lambda_{2,1}$ -labeling of G . Then $m_i = m_h$ for $1 \leq i, h \leq r + 1$.*

Proof. Let T be the set of integers in the interval $[0, r + 2]$. Then for every integer x , $1 \leq x \leq r + 1$, there are exactly r elements in T which differ from x by at least 2. Thus, the distance conditions require the labels of the neighbors of each vertex v with label $L(v) = x$ to be precisely the elements of $S_x = \{w \in T \mid |x - w| \geq 2\}$. So, for every $y \in S_x$, we have $m_x \leq m_y$. But if $y \in S_x$ where $1 \leq y \leq r + 1$, then $x \in S_y$, implying $m_y \leq m_x$. Hence $m_h = m_{2+h} = m_{3+h} = m_{h+1}$ for $1 \leq h \leq r - 2$, giving the result. \square

Now let L be a $(2, 1, r)$ -optimal labeling of r -regular graph G . We define $M(\alpha, \beta)$ to be the set of vertices in $V(G)$ which have label α and which are adjacent to some vertex with label β , and we denote the cardinality of $M(\alpha, \beta)$ by $m(\alpha, \beta)$. Noting that, for v such that $L(v) = 0$, exactly one element i of the set $\{2, 3, 4, \dots, r + 2\}$ is not represented among the labels of the neighbors of v , we define $M(0, i^*)$ to be the collection of vertices which are labeled 0 and which are adjacent to no vertices labeled i . For $i \in \{0, 1, 2, 3, 4, \dots, r\}$, we define $M(r + 2, i^*)$ analogously, and we denote the cardinalities of $M(0, i^*)$ and $M(r + 2, i^*)$ by $m(0, i^*)$ and $m(r + 2, i^*)$, respectively. For fixed h , $2 \leq h \leq r + 1$, there is a one-to-one correspondence between M_h and $M_0 - M(0, h^*)$, implying that $m_h = m_0 - m(0, h^*)$. So, by Theorem 5.1, it follows that $m(0, 2^*) = m(0, 3^*) = \dots = m(0, r + 1^*)$. Similarly, for $1 \leq i \leq r$, $m(r + 2, 1^*) = m(r + 2, 2^*) = \dots = m(r + 2, r^*)$.

Since $m(0, r + 2) = m(r + 2, 0)$, we have

$$(1) \quad \sum_{i=2}^{r+1} m(0, i^*) = m(0, r + 2) = m(r + 2, 0) = \sum_{i=1}^r m(r + 2, i^*).$$

But for every i , $2 \leq i \leq r + 1$, each vertex in M_i is adjacent to some vertex in M_0 , implying $m(0, i) = m_i$. Thus, since $m(i, 0) = m(0, i)$, we have $m_0 = m(0, i) + m(0, i^*) = m_i + m(0, i^*)$, giving $m_0 - m_i = m(0, i^*)$. But $m_2 = m_3 = \dots = m_{r+1}$ by Theorem 5.1, so $m(0, i^*) = m_0 - m_2$ for every i , $2 \leq i \leq r + 1$.

Similarly, $m(r + 2, i^*) = m_{r+2} - m_2$, which, by (1), implies

$$\sum_{i=2}^{r+1} (m_0 - m_2) = \sum_{i=1}^r (m_{r+2} - m_2).$$

This gives the following theorem.

THEOREM 5.2. *Let $G \in \Gamma(2, 1, r)$ and let L be a $\lambda_{2,1}$ -labeling of G . Then $m_0 = m_{r+2}$.*

For fixed h , $2 \leq h \leq r + 1$, there is a one-to-one correspondence between M_h and $M_0 - M(0, h^*)$, implying that $m_h = m_0 - m(0, h^*)$ (and likewise, $m_h = m_{r+2} - m(r + 2, h^*)$ for $1 \leq h \leq r$). So, by Theorems 5.1, 5.2, and (1), it follows that

$$\begin{aligned} m(0, 2^*) &= m(0, 3^*) = \dots = m(0, r + 1^*) \\ &= m(r + 2, 1^*) = m(r + 2, 2^*) = \dots = m(r + 2, r^*). \end{aligned}$$

We use this result to establish the next theorem.

THEOREM 5.3. *Let $G \in \Gamma(2, 1, r)$ and let L be a $\lambda_{2,1}$ -labeling of G . Then $|V(G)| = (r + 3)m(0, r + 2^*) + (r^2 + 2r - 1)m(0, 2^*)$.*

Proof. By Theorems 5.1 and 5.2, $|V(G)| = \sum_{i=0}^{r+2} m_i = 2m_0 + (r + 1)m_2$. Since

$$m_0 = \sum_{i=2}^{r+2} m(0, i^*) = rm(0, 2^*) + m(0, r + 2^*)$$

and

$$\begin{aligned} m_2 &= m_0 - m(0, 2^*) = -m(0, 2^*) + \sum_{i=2}^{r+2} m(0, i^*) = \sum_{i=3}^{r+2} m(0, i^*) \\ &= m(0, r + 2^*) + \sum_{i=3}^{r+1} m(0, i^*) = m(0, r + 2)^* + (r - 1)m(0, 2^*), \end{aligned}$$

the result now follows via straightforward algebra. \square

Since $m(0, 2^*)$ and $m(0, r + 2^*)$ must be nonnegative, we observe that the smallest $(2, 1, r)$ -optimal graph has order at least $r + 3$. As noted in the preceding section, this bound is achieved by the unique r -regular graph on $r + 3$ vertices: $\overline{C_{r+3}}$, which is cyclically optimal. If $G \in \Gamma(2, 1, r)$ is acyclically optimal, then every optimal labeling of G has $m(0, r + 2) \geq 1$, which in turn implies that $m(0, 2^*) \geq 1$. Thus, the smallest $(2, 1, r)$ -acyclically optimal graph has order at least $r^2 + 2r - 1$. We produce a $(2, 1, r)$ -acyclically optimal graph $G_a(r)$ on $r^2 + 2r - 1$ vertices as follows: noting that $m_0 = m_{r+2} = r$ and $m_1 = m_2 = \dots = m_{r+1} = r - 1$ under an acyclically optimal labeling of G , we define

$$\begin{aligned} M_i &= \{v_i^h, 1 \leq h \leq r - 1\} \text{ for } 1 \leq i \leq r + 1, \text{ and} \\ M_i &= \{v_i^p, 1 \leq p \leq r\} \text{ for } i = 0, r + 2. \end{aligned}$$

Let $P_0 = \{S_1, S_2, S_3, \dots, S_r\}$, where S_i is lexicographically the i th subset size $r - 1$ of $\{2, 3, 4, \dots, r + 1\}$. Similarly, let $P_{r+2} = \{T_1, T_2, T_3, \dots, T_r\}$, where T_i is lexi-

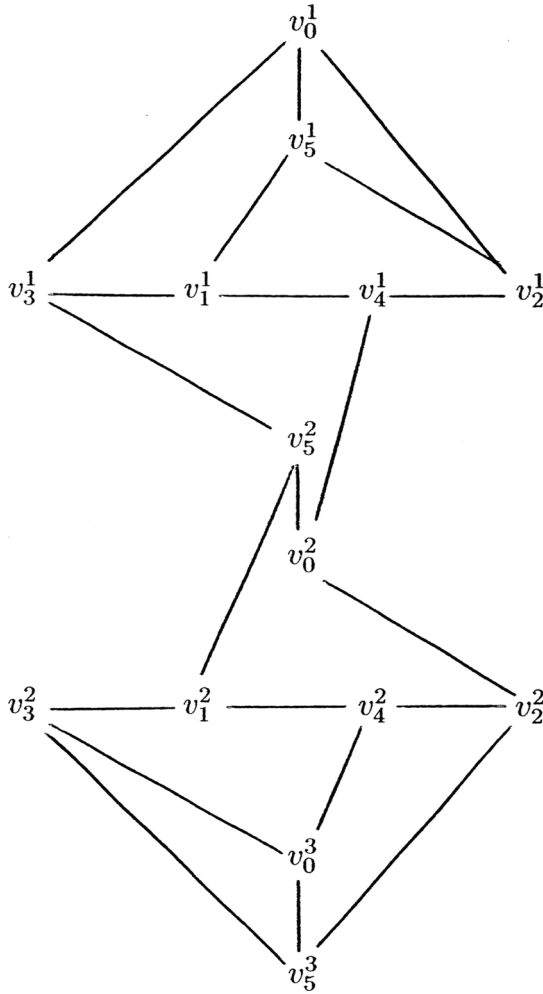


FIG. 5.1. A $\lambda_{2,1}$ -labeling of a $(2, 1, 3)$ -acyclically optimal graph on 14 vertices.

graphically the i th subset size $r - 1$ of $\{1, 2, 3, \dots, r\}$. We define the edges of G as follows:

1. For $1 \leq p \leq r$, $\{v_0^p, v_{r+2}^p\} \in E(G)$.
2. For $1 \leq h \leq r - 1$, $\{v_s^h, v_t^h\} \in E(G)$ if and only if $|s - t| \geq 2$, $1 \leq s, t \leq r + 1$.
3. For $1 \leq p \leq r$, $2 \leq i \leq r + 1$ and $1 \leq h \leq r - 1$, $\{v_0^p, v_i^h\} \in E(G)$ if and only if S_p contains i and there are exactly $h - 1$ sets S_1, S_2, \dots, S_{p-1} which contain i .
4. For $1 \leq p \leq r$, $1 \leq i \leq r$ and $1 \leq h \leq r - 1$, $\{v_{r+2}^p, v_i^h\} \in E(G)$ if and only if T_p contains i and there are exactly $h - 1$ sets T_1, T_2, \dots, T_{p-1} which contain i .

Then the labeling L given by $L(v_z^i) = z$ is a $(2, 1, r)$ -acyclically optimal labeling of G . In Figure 5.1, we illustrate a $\lambda_{2,1}$ -labeling of a $(2, 1, 3)$ -acyclically optimal graph on 14 vertices.

The existence of $(2, 1, r)$ -optimal graphs on $r + 3$ vertices and on $r^2 + 2r - 1$ vertices leads to the following theorem.

THEOREM 5.4. *For $x, y \in Z^+$, there exists a $(2, 1, r)$ -optimal graph on $x(r+3) + y(r^2 + 2r - 1)$ vertices.*

COROLLARY 5.5. *If r is even, then for all $n \geq (r+2)(r^2 + 2r - 2)$, there exists a $(2, 1, r)$ -optimal graph on n vertices. If r is odd, then for all $n \geq \frac{(r-5)(r^2+2r-3)}{4}$, there exists a $(2, 1, r)$ -optimal graph on $2n$ vertices.*

Proof. If r is even, then $\gcd(r+3, r^2 + 2r - 1) = 1$, implying every integer greater than or equal to $(r+2)(r^2 + 2r - 2)$ can be written as a linear combination of $r+3$ and $r^2 + 2r - 1$ with nonnegative coefficients. If r is odd, then $\gcd(r+3, r^2 + 2r - 1) = 2$, giving the result by a similar argument. \square

Although the $(2, 1, r)$ -cyclically optimal graph on $r+3$ vertices is unique, such is not the case for $(2, 1, r)$ -acyclically optimal graphs on $r^2 + 2r - 1$ vertices. However, each $(2, 1, r)$ -acyclically optimal graph on $r^2 + 2r - 1$ vertices, through a sequence of edge transpositions similar to that described in the preceding section, yields a graph isomorphic to $G_a(r)$. Extending this argument gives the following theorem.

THEOREM 5.6. *Every $(2, 1, r)$ -optimal graph yields, through a sequence of edge transpositions, a graph isomorphic to a sum of copies of $\overline{C_{r+3}}$ and $G_a(r)$.*

6. On prisms. In this section, we apply our results on optimality to a special class of 3-regular graphs known as prisms. For $n \geq 3$, the n -prism, denoted $\text{Pr}(n)$, is the graph consisting precisely of two disjoint n -cycles v_0, v_1, \dots, v_{n-1} and w_0, w_1, \dots, w_{n-1} and edges $\{v_i, w_i\}$ for $0 \leq i \leq n-1$. The two cycles shall be called the inner and outer cycles, respectively. We point out that $\text{Pr}(n)$ is isomorphic to $C_n \times P_2$. We also note that it will be convenient to exhibit a labeling of $\text{Pr}(n)$ in the form of a $2 \times n$ array, where the entries in the top row of the array correspond to the labels of the vertices of the outer cycle and the entries in the bottom row correspond to the labels of the vertices of the inner cycle.

In [14], Jha et al. proved the following theorem.

THEOREM 6.1. *Let $n \geq 3$. Then*

$$\lambda_{2,1}(\text{Pr}(n)) \begin{cases} = 5 & \text{if } n \equiv 0 \pmod{3}, \\ \leq 6 & \text{if } n \not\equiv 0 \pmod{3}. \end{cases}$$

We refine this theorem as follows (and are informed that an alternative proof will appear in [15]).

THEOREM 6.2. *Let $n \geq 3$. Then*

$$\lambda_{2,1}(\text{Pr}(n)) = \begin{cases} 5 & \text{if } n \equiv 0 \pmod{3}, \\ 6 & \text{if } n \not\equiv 0 \pmod{3}. \end{cases}$$

Proof. By Theorem 6.1, it suffices to show that $\lambda_{2,1}(\text{Pr}(n)) > 5$ for $n \not\equiv 0 \pmod{3}$. Suppose to the contrary that there exists an n , $n \equiv 1, 2 \pmod{3}$, such that $\lambda_{2,1}(\text{Pr}(n)) = 5$. Let L be a $\lambda_{2,1}$ -labeling of $\text{Pr}(n)$. Since the order of $\text{Pr}(n)$ is $2n$, we observe that $|V(\text{Pr}(n))| \equiv 2, 4 \pmod{6}$, which by Theorem 4.3 implies that L is acyclic. Thus, by the discussion following Theorem 5.1, $m(0, i^*) \geq 1$, $2 \leq i \leq 4$, implying $m(0, 3^*) \geq 1$.

With no loss of generality, let $a_0, a_1, a_2, b_0, b_1, b_2$ be vertices in $V(\text{Pr}(n))$ such that $a_1 \in M(0, 3^*)$. Then the neighbors of a_1 , namely, a_0, a_2 , and b_1 , receive the labels 2, 4, and 5 under L (not necessarily respectively). If a_0 or b_1 receives the label 2, then by virtue of the 4-cycle $\langle a_0, a_1, b_1, b_0 \rangle$, $L(b_0) \geq 6$, contradicting the optimality of L . If a_2 receives the label 2, then by virtue of the 4-cycle $\langle a_1, a_2, b_2, b_1 \rangle$, $L(b_2) \geq 6$, another contradiction. \square

For $\frac{j}{k} > r$, if $\text{Pr}(n) \in \Gamma(j, k, 3)$, then by Theorem 3.1, $\text{Pr}(n)$ is bipartite (implying that n is even) and $2n \equiv 0 \pmod{6}$. Hence $n \equiv 0 \pmod{6}$, a condition which is easily seen to be sufficient for optimality by labeling the vertices of the inner cycle $0, 2, 4, 0, 2, 4, \dots, 0, 2, 4$ and the vertices of the outer cycle $3, 5, 1, 3, 5, 1, \dots, 3, 5, 1$.

If $\text{Pr}(n) \in \Gamma(1, 1, 3)$, then by Theorem 4.3, $|V(\text{Pr}(n))| \equiv 0 \pmod{4}$. Hence, n is even, so $\text{Pr}(n)$ is bipartite. By Theorem 4.5, $|V(\text{Pr}(n))| \equiv 0 \pmod{8}$, implying the necessary condition $n \equiv 0 \pmod{4}$. However, this condition is also sufficient for the $(1, 1, 3)$ -optimality of $\text{Pr}(n)$, as shown in the definitive calculation of $\lambda_{1,1}(\text{Pr}(n))$, given below.

THEOREM 6.3. *Let $n \geq 3$. Then*

$$\lambda_{1,1}(\text{Pr}(n)) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod{4}, \\ 5 & \text{if } n = 3, 6, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. For $n \equiv 0 \pmod{4}$, consider the array A_1 , which represents a $\lambda_{1,1}$ -labeling of $\text{Pr}(4)$:

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1. \end{array}$$

Then if $n = 4m$, an optimal $(1, 1, 3)$ -labeling of $\text{Pr}(n)$ is demonstrable by concatenating m copies of A_1 like so:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & \dots & 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & \dots & 2 & 3 & 0 & 1. \end{array}$$

For $n = 3, 6$, consider $n = 3$. Then $\text{Pr}(3)$ has diameter two, which implies (by the distance conditions) that no two vertices may be assigned the same label. It is then an easy matter to show the existence of an $L(1, 1)$ -labeling of $\text{Pr}(3)$ with span equal to the lower bound 5. If $n = 6$, then the converse of part *a* implies that $\lambda_{1,1}(\text{Pr}(6)) \geq 4$. But if $\lambda_{1,1}(\text{Pr}(6)) = 4$, the pigeon-hole principle implies the existence of a label with multiplicity 3. The distance constraints, however, imply that no label may have multiplicity 3. Thus, $\lambda_{1,1}(\text{Pr}(6)) = 5$, as demonstrated by the following labeling:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 0 & 1. \end{array}$$

In the final case, we note that n is not a multiple of 4, implying that $\lambda_{1,1}(\text{Pr}(n)) \geq 4$. It thus suffices to show the existence of an $L(1, 1)$ -labeling with span 4. To that end, consider the array A_2 , which represents a $\lambda_{1,1}$ -labeling of $\text{Pr}(5)$:

$$\begin{array}{cccc} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1. \end{array}$$

Then, since any integer n , $n > 11$ and n not divisible by 4, can be written $4\alpha + 5\beta$ for some $\alpha \geq 0$ and $\beta \geq 1$, we can demonstrate an $L(1, 1)$ -labeling with span 4 by the concatenating α copies of A_1 and β copies of A_2 .

In the remaining cases $n = 7$ and $n = 11$, we demonstrate $L(1, 1)$ -labelings with span 4:

$$\begin{array}{cccccc} 0 & 4 & 1 & 0 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 2 & 0 & 3 \end{array}$$

and

$$\begin{array}{cccccccc} 0 & 2 & 3 & 1 & 0 & 2 & 4 & 1 & 0 & 3 & 4 \\ 3 & 1 & 0 & 4 & 3 & 1 & 0 & 2 & 4 & 1 & 2. \end{array} \quad \square$$

Let the Cartesian product of the infinite path P_∞ and P_2 be denoted by $\text{Pr}(\infty)$. By an approach similar to the one used in the proof of Theorem 2.3, we may establish that $\lambda_{j,k}(\text{Pr}(\infty)) \leq \lambda_{j,k}(\text{Pr}(n))$ for all $n \geq 3$; furthermore, it can be shown that $\lambda_{1,1}(\text{Pr}(\infty)) = 3$ and $\lambda_{2,1}(\text{Pr}(\infty)) = 5$ and that $\text{Pr}(\infty)$ is both $(1, 1, 3)$ - and $(2, 1, 3)$ -cyclically optimal. (By Theorem 4.1, $\text{Pr}(\infty)$ is $(3, 1, 3)$ -optimal.) Finally, analysis analogous to that employed in the proof of Theorem 6.2 reveals that all optimal $(2, 1, 3)$ -labelings of $\text{Pr}(\infty)$ are cyclic with even labels appearing along one copy of P_∞ and odd labels along the other.

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