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Two-dimensional descent through a compressible atmosphere: Sequential deceleration of an unpowered load

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Abstract – Equations, based on Rayleigh's drag law valid for high Reynolds number, are derived for two-dimensional motion through a compressible atmosphere in isentropic equilibrium, such as characterizes the Earth's troposphere. Solutions yield horizontal and vertical displacement, velocity, and acceleration as a function of altitude and ground-level temperature. An exact analytical solution to the equations linearized in the aero-thermodynamic parameter is given; in general the equations must be solved numerically. The theory, applied to the unpowered fall of a large aircraft stabilized to flat descent by symmetrical, sequential deployment of horizontal and vertical decelerators, shows that such an aircraft can be brought down with mean peak deployment and impact decelerations below $10g$.

Introduction. – The air resistance (drag force) on an object descending through an atmosphere depends on the air density, square of the relative speed, effective area presented to the air stream, and drag coefficient [1]. The air density in turn is a function of altitude and air temperature. In an atmosphere in isentropic equilibrium, such as characterizes the Earth's troposphere (depth 8–16 km from poles to tropics) [2], the density varies adiabatically with altitude. The drag coefficient is largely independent of size, but depends weakly on Reynolds number and sensitively on shape and origin (*i.e.* from pressure or friction). In this paper I derive and investigate a set of coupled nonlinear differential equations characterizing the two-dimensional descent through an ideal-gas atmosphere under adiabatic conditions of a composite object subject to a quadratic drag force as first proposed by Rayleigh for high Reynolds number ($Re > 1000$). The coupling of horizontal and vertical motions lead to results that can differ significantly from one-dimensional applications of Rayleigh's equation for drag in an incompressible fluid.

The theory developed here facilitates realistic modeling of the impact of temperature and density variations on air drag, serves as a model for extension to more general polytropic atmospheres, and permits exact analysis of the controlled descent of fragile loads, a topic of vital concern to space agencies, cargo transporters, and general aviation. In regard to the latter, in particular, I illustrate the significance of the theory by demonstrating how a large passenger airliner, having suffered total loss of power, may be brought to ground by means of a sequentially released parachute-assisted descent with impact deceleration below $10g$, where $g = 9.8\,\mathrm{m/s^2}$ is the acceleration of gravity near the Earth's surface.

The trend in design of modern commercial aircraft, driven in part by rising costs of fuel, construction materials, and labor, is to larger, heavier planes that transport ever greater numbers of passengers. Aerodynamicists now routinely contemplate design models capable of carrying 800 or more people [3]. Although air travel is presently considered very safe, no human-made machine is 100% reliable, and it is therefore certain that at least one of these airplanes would eventually fail in service with a huge number of fatalities. It is consequently of major importance to investigate how the laws of physics may be used to avert such a catastrophe.

The maximum acceleration that a human can endure has long been of practical interest to organizations concerned with high-speed transport. Estimates have ranged from about $10g$ to $100g$, depending on duration and orientation of impact. Particularly striking was the case of racing driver David Purley who survived a crash estimated to have produced $179g$ [4]. A comprehensive

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study of human impact tolerance prepared for the Insurance Institute for Highway Safety [5] reported, among other findings, that $350g$ for 2.5–3.0 ms was the approximate survival limit of children under age 8 subject to head impacts. From such data it seems likely that a few hundred g over a period of a few seconds would be a liberal upper limit to human impact tolerance under most circumstances.

The idea of protecting an entire aircraft, rather than individual persons, with a parachute, unusual as it may seem, has in fact been implemented commercially since 1980 for small craft with maximum weights in the range of about 270–1410 kg and deploy speeds of about 65–85 m/s [6]. For large general aviation aircraft, the greater weights, speeds, and altitudes are believed to make in-air recovery virtually impossible. The significant practical finding of this paper is that, in-air recovery of large general-aviation aircraft should be aerodynamically feasible with decelerators of a size that currently exist and without necessarily requiring new materials.

Dynamics of freefall through a compressible atmosphere. – Newton's 2nd law of motion applied to an object of mass m, plan area S, and drag coefficient C descending with velocity $\mathbf{v}$ through an atmosphere of density ρ with quadratic drag law takes the form

$$m\mathrm{d}\mathbf{v}/\mathrm{d}t + \tfrac{1}{2}\rho C S\,(\mathbf{v}\cdot\mathbf{n})\,\mathbf{v} = m\mathbf{g}, \tag{1}$$

where $\mathbf{n}$ is the outward normal to the surface facing the air stream and $\mathbf{g}$ is oriented downward and assumed constant in this analysis. The plan area (or planform) S is the maximum area projected normally onto a plane. For an object of total mass m comprising separate but attached plates (fig. 1) contributing drag independently [7] with surfaces either perpendicular to the horizontal (x-axis) or facing downward (y-axis), the components of eq. (1) can be expressed as a set of coupled first-order nonlinear equations:

$$\begin{aligned} &\mathrm{d}v_x/\mathrm{d}t + g^{-1}\left(\beta_x^2 v_x^2 + \beta_y^2 v_x v_y\right) = 0,\\ &\mathrm{d}v_y/\mathrm{d}t + g^{-1}\left(\beta_y^2 v_y^2 + \beta_x^2 v_x v_y\right) = g, \end{aligned} \tag{2}$$

where

$$v_x = \mathrm{d}x/\mathrm{d}t = v\cos\theta\,, \quad v_y = \mathrm{d}y/\mathrm{d}t = v\sin\theta. \tag{3}$$

The incidence θ is the angle the air stream makes with the horizontal. Drag parameters of the x- and y-oriented plates are defined by

$$\beta_x^2 = \frac{g\rho}{2m}\sum_{x\text{-plates}\,i} C_i S_i, \quad \beta_y^2 = \frac{g\rho}{2m}\sum_{y\text{-plates}\,i} C_i S_i. \tag{4}$$

In a plate model of a falling unpowered aircraft with vertical and horizontal decelerators (fig. 1), air drag is due primarily to pressure, rather than friction, on the airfoil (wings), vertical parachute(s) (vp), and horizontal (or drogue) parachute (hp), each component characterized

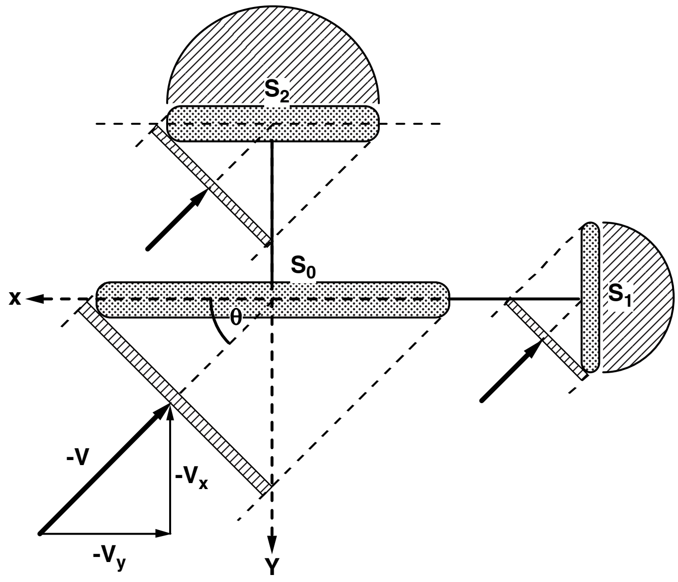

Fig. 1: Schematic diagram of an airfoil with horizontal and vertical decelerators, modeled as plates with respective plan areas S_0, S_1, S_2, moving relative to the air stream with velocity $\mathbf{v}$ and incidence θ (as seen from the rest frame of the airfoil).

aerodynamically as a plate of appropriate planform and drag coefficient. For such a configuration, eq. (4) takes the simplified form

$$\begin{aligned} &\beta_x^2 = g\rho C_{\mathrm{hp}} S_{\mathrm{hp}}/2m,\\ &\beta_y^2 = g\rho\left(C_{\mathrm{wings}} S_{\mathrm{wings}} + C_{\mathrm{vp}} S_{\mathrm{vp}}\right)/2m. \end{aligned} \tag{5}$$

In an atmosphere in isentropic equilibrium, the air density varies adiabatically with altitude $y' = h - y$, where h is the initial height according to the expression [8]

$$\begin{aligned} \rho\,(y')_{\mathrm{air}} = {} & \rho_0\left(1-\gamma^{-1}(\gamma-1)\,M_{\mathrm{air}} g y'/RT_0\right)^{\frac{1}{\gamma-1}} \sim \\ & \rho_0\,(1 - M_{\mathrm{air}} g y'/\gamma R T_0). \end{aligned} \tag{6}$$

ρ_0 is the air density at ground level ($y' = 0$), T_0 is the ground-level temperature, γ is the ratio of heat capacity at constant pressure to heat capacity at constant volume ($\sim$1.4 for an ideal diatomic gas), M_{air} is the mean molar mass of air ($\sim$28.97 g), and $R = 8.314$ J/K is the universal gas constant. Equation (6) follows from combined application of the ideal-gas equation of state ($P = \rho RT/M$), barometric equation ($\mathrm{d}P/\mathrm{d}y' = -Mg\rho/RT$) for pressure P, and adiabatic lapse rate [$\mathrm{d}T/\mathrm{d}y' = -(Mg(\gamma-1)/\gamma R)$], which amounts to $-10\,^{\circ}$C/km in dry air [9]. The second relation in eq. (6) is obtained by truncation of the Taylor series expansion to first order in y'/h_{atm}, where $h_{\mathrm{atm}} \equiv \gamma R T_0/(\gamma-1) M_{\mathrm{air}}\, g$ is the adiabatic height of the atmosphere ($\sim$28 km for $T_0 \sim 273$ K) at which pressure would fall to zero if air temperature continued to fall linearly with altitude.

Substitution of eq. (6) into eq. (5) and elimination of time derivatives by setting $\mathrm{d}/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}t)(\mathrm{d}/\mathrm{d}y) = v_y(\mathrm{d}/\mathrm{d}y)$ leads to

$$\begin{aligned} &v_y \mathrm{d}v_x/\mathrm{d}y + g^{-1}\left(\beta_{x0}^2 v_x^2 + \beta_{y0}^2 v_x v_y\right) \\ &\times (1 - M_{\mathrm{air}} g(\gamma-1)(h - y/\gamma R T_0))^{\frac{1}{\gamma-1}} = 0, \\ &v_y \mathrm{d}v_y/\mathrm{d}y + g^{-1}\left(\beta_{y0}^2 v_y^2 + \beta_{x0}^2 v_x v_y\right) \\ &\times (1 - M_{\mathrm{air}} g(\gamma-1)(h-y)/\gamma R T_0)^{\frac{1}{\gamma-1}} = g, \end{aligned} \tag{7}$$

where (β_{x0}, β_{y0}) are the ground-level drag parameters defined by eq. (5) for $\rho = \rho_0$. Finally, it is useful for modeling and computation to cast the eq. (7) into dimensionless form:

$$\begin{aligned} &V_y \frac{\mathrm{d}V_x}{\mathrm{d}Y} + \left(\alpha^2 V_x^2 + V_x V_y\right)(1 - \kappa(H-Y))^{\frac{1}{\gamma-1}} = 0, \\ &V_y \frac{\mathrm{d}V_y}{\mathrm{d}Y} + \left(V_y^2 + \alpha^2 V_x V_y\right)(1 - \kappa(H-Y))^{\frac{1}{\gamma-1}} = 1, \end{aligned} \tag{8}$$

with $(H \geqslant Y \geqslant 0)$ by scaling velocity, time, and displacement

$$\begin{aligned} V_x &= \beta_{y0} v_x/g, \quad V_y = \beta_{y0} v_y/g, \quad T = \beta_{y0} t, \\ \alpha &= \beta_{x0}/\beta_{y0}, \quad Y = \beta_{y0}^2 y/g, \quad H = \beta_{y0}^2 h/g, \end{aligned} \tag{9}$$

and defining the aero-thermodynamic parameter

$$\kappa = M_{\mathrm{air}} g^2 (\gamma - 1)/\gamma \beta_{y0}^2 R T_0 = h_{\mathrm{v}}/h_{\mathrm{atm}}, \tag{10}$$

in which $h_{\mathrm{v}} \equiv g/\beta_{y0}^2$ is the distance fallen from rest to $\sim 93\%$ of vertical terminal velocity (*i.e.* for $t = \beta_{y0}^{-1}$) in a homogeneous atmosphere. The scaled components of acceleration are then

$$A_x = \mathrm{d}V_x/\mathrm{d}T = a_x/g, \quad A_y = \mathrm{d}V_y/\mathrm{d}T = a_y/g. \tag{11}$$

In full generality, eqs. (8) require numerical solution. They can be solved analytically, however, for several important special cases.

A) *Stationary solutions.*

Setting $\mathrm{d}v_x/\mathrm{d}t = \mathrm{d}v_y/\mathrm{d}t = 0$ in eqs. (2) leads to stationary solutions

$$v_{s,x} = 0, \qquad v_{s,y} = g/\beta_y. \tag{12}$$

In the special case of 1D drag in a homogeneous fluid medium, the stationary velocities are also the terminal velocities defined by the limit $t \to \infty$. In the general case, however, stationary and terminal vertical velocities may differ, as shown by the stationary solutions of eqs. (8):

$$V_{\mathrm{s},x} = 0, \qquad V_{\mathrm{s},y} = (1 - \kappa(H-Y))^{\frac{-1}{\gamma-1}}. \tag{13}$$

Because of the restriction $Y \leqslant H$, one cannot meaningfully take the unbounded limit of t or y, and terminal velocity $V_{t,y}$ is then understood to mean $V_{s,y}(H) = 1$ or equivalently $v_{s,y} = g/\beta_{y0}$.

B) *Vertical descent through a linear compressible atmosphere.*

In the absence of a horizontal component $(V_x = 0)$, eq. (8) reduces to a linear, first-order differential equation in $V_y^2(Y')$ (with $Y' = H - Y$ the reduced altitude), which takes the form

$$\begin{aligned} &\mathrm{d}V_y^2/\mathrm{d}Y' - 2(1 - \kappa' Y') V_y^2 = -2, \\ &\kappa' = (\gamma - 1)\kappa = M_{\mathrm{air}} g^2/\gamma R T_0 \beta_{y0}^2, \end{aligned} \tag{14}$$

upon substitution of the linear approximation in eq. (6) to the variation in air density. Equation (14) can be solved exactly by means of an integrating factor [10] to yield

$$\begin{aligned} &V_y^2(Y') = V_{0y}^2 e^{\kappa'\left(H^2 - Y'^2\right) - 2\left(H - Y'\right)} \\ &+ 2e^{-1/\kappa'} \kappa'^{-1/2} e^{-\left(\kappa' Y'^2 - 2Y'\right)} \int_{\sqrt{\kappa'}(Y' - \kappa'^{-1})}^{\sqrt{\kappa'}\left(H - \kappa'^{-1}\right)} e^{u^2} \mathrm{d}u, \end{aligned} \tag{15}$$

with initial condition $V_y(H) = V_{y0}$. The scaled acceleration is then

$$A_y = -\tfrac{1}{2}\mathrm{d}V_y^2/\mathrm{d}Y' = 1 - (1 - \kappa' Y') V_y^2. \tag{16}$$

The relation between velocity and time must be obtained through integration $T = \int_Y^H \frac{\mathrm{d}u}{V_y(u)}$.

C) *1D horizontal and vertical descent through a homogeneous atmosphere.*

Uncoupling x and y components of eq. (2) and setting $\kappa = 0$ leads to integrable equations whose solutions with initial velocities v_{0x}, v_{0y} are summarized in table 1 for both scaled and dimensioned variables. Figure 2 shows the variation in velocities V_x, V_y with vertical displacement Y for the exact 2D theory (eq. (8)) and decoupled 1D approximation (table 1) for the unpowered descent of an aircraft cruising horizontally with parameters pertinent to applications in the following section. Two notable features are a) the faster decline of horizontal velocity with displacement in the 2D theory, and b) the rise of vertical velocity above the terminal limit $V_{\mathrm{t},y} = 1$, with subsequent decline to $V_{\mathrm{t},y} = 1$ in the 2D theory. Also shown in fig. 2 are 2D velocity profiles in the case of zero horizontal drag $(\alpha = 0)$. In marked contrast to the 1D case for which there would be no horizontal deceleration, the coupling of V_x and V_y in the exact 2D analysis generates a horizontal deceleration comparable to that achievable with a drogue parachute.

Stabilization and recovery of general aviation aircraft. – The theory of the previous section permits investigation of protocols to bring to ground an unpowered general aviation aircraft with decelerations at all stages within a range of passenger survivability, *i.e.* $\leqslant\sim 10g$. For illustrative purposes I consider a plane comparable to a B-747-100 Jumbo Jet, whose relevant parameters are given in table 2.

For in-air recovery of a crippled B-747 a horizontal parachute is first deployed from the rear to reduce cruising

Table 1: Solutions to uncoupled equations of motion for homogeneous density.

Component	Scaled variables	Dimensioned variables
Horizontal velocity	$V_x(T) = \frac{V_{0x}}{V_{0x}T+1}$	$v_x(t) = \frac{v_{0x}}{(\beta_y^2/g)v_{0x}t+1}$
	$V_x(\mathrm{X}) = V_{0x}e^{-X}$	$v_x(\mathrm{x}) = v_{0x}e^{-\beta_x^2 x/g}$
Horizontal acceleration	$A_x(t) = -(\frac{V_{0x}}{V_{0x}T+1})^2$	$\frac{a_x(t)}{g} = -(\frac{\beta_x v_{0x}/g}{(\beta_x^2/g)v_{0x}t+1})^2$
Horizontal displacement	$X(t) = \ln[V_{0x}T+1]$	$x(t) = \frac{g}{\beta_x^2}\ln[(\beta_x^2/g)v_{0x}t+1]$
Vertical velocity	$V_y(T) = \frac{\tanh(T)+V_{0y}}{1+\tanh(T)}$	$v_y(t) = \frac{(g/\beta_y)\tanh(\beta_y t)+v_{0y}}{1+(\beta_y v_{0y}/g)\tanh(\beta_y t)}$
	$V_y(Y) = [1-(1-V_{0y}^2)e^{-2Y}]^{\frac{1}{2}}$	$v_y(y) = \frac{g}{\beta_y}[1-(1-(\frac{\beta_y v_{0y}}{g})^2)e^{-2\beta_y^2 y/g}]^{\frac{1}{2}}$
Vertical acceleration	$A_y(T) = \frac{1-V_{0y}^2}{(\cosh(T)+V_{0y}\sinh(T))^2}$	$\frac{a_y(t)}{g} = \frac{1-(\beta_y v_{0y}/g)^2}{(\cosh(\beta_y t)+(\beta_y v_{0y}/g)\sinh(\beta_y t))^2}$
Vertical displacement	$Y(T)=\ln[\cosh(T)+V_{0y}\sinh(T)$	$y(t) = \frac{g}{\beta_y^2}\ln[\cosh(\beta_y t)+(\beta_y v_{0y}/g)\sinh(\beta_y t)]$

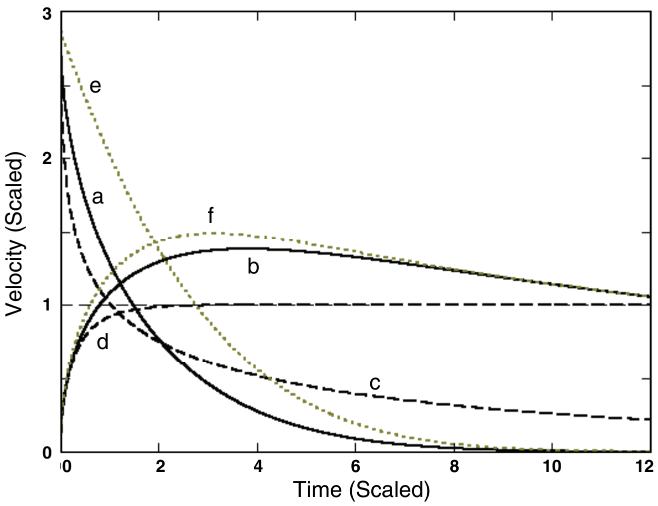

Fig. 2: (Colour on-line) Plot of time variation of scaled horizontal (a, c, e) and vertical (b, d, f) components of velocity for an unpowered B747-100 with air drag provided by airfoil and drogue parachute calculated by the exact 2D theory (solid line) and uncoupled 1D theory (dashed line); airfoil only, calculated by the exact 2D theory (dotted line). Initial height (km) is $h=10$; initial velocity components (m/s) are $v_x = 250$, $v_y = 1$. Plan areas (m^2) are $S_{\mathrm{airfoil}} = 511$, $S_{\mathrm{drogue}} = 182.4$. Drag parameters (s^{-1}) are $\beta_{x0} = 0.0722$, $\beta_{y0} = 0.1125$ with aero-thermodynamic parameter $\kappa = 0.028$. The dashed line at ordinate 1 marks the terminal vertical velocity.

speed from $\sim$250 m/s to $\sim$50 m/s followed by symmetrical deployment from ports along the upper surface of the fuselage of one or more vertical parachutes to decelerate the rate of descent to a survivable terminal velocity $v_{\mathrm{t},y}$. If L_c is the impact deceleration length at the ground, then the objective of the rescue protocol is to insure an impact deceleration,

$$a_c/g = v_{\mathrm{t},y}^2/2gL_c = g/2\beta_{y0}^2 L_c, \qquad (17)$$

below 10.

Table 2: Characteristics of B-747-100 Jumbo Jet.

Length of fuselage	74.2 m
Diameter of fuselage	6.5 m
Wingspan	68.4 m
Wing area	511 m^2
Empty mass	162386 kg
Loaded mass	333390 kg
Aspect Ratio (AR)	7.0
Mininum (frictional) drag coefficient C_d	0.031

Vertical decelerators comparable to the commercially available G-11 cargo parachute [11] of nominal radius $R = 15.24$ m, surface area $S_p = \pi R^2 = 729.7$ m^2, and mass 113.4 kg would suffice, with a corresponding parachute of radius $R/2$ for the drogue. The manufacturer packages these parachutes in clusters up to 8. Sequential deployment symmetrically over the fuselage makes it possible to reduce vertical impact with the ground to a level below that of individual military parachutists (10–15g) [12].

The drag coefficient of a parachute C_p depends on shape and venting, and a spread of values can be found in the literature [13] ranging from about 1.3 to 2.4 depending on the mode of descent. In the following analysis, I adopt $C_p = 1.5$. The drag coefficient of a plate (the airfoil) of aspect ratio 7.0 at high Reynolds number is approximately $C_{\mathrm{plate}} = 1.3$ [14]. Given the maximum take-off mass in table 2 and STP ground-level values for air density (1.294 kg/m^3) and temperature (273 K), the drag parameters for the horizontal and vertical decelerators become $\beta_{x0} = 0.0722$, $\beta_{y0} = \sqrt{0.0126 + 0.0208 n_p}$, where n_p is the number of vertical parachutes deployed. Since $C_d/C_p \sim 0.021$, we can ignore the contribution of the aircraft's frictional drag when treating horizontal deceleration.

Upon substitution of the preceding parameters into eqs. (8), one finds that a B-747 would decelerate to

Table 3: Parachute-assisted descent of a B747-100 Aircraft.

Action	n_p	β_x (s^{-1})	β_y (s^{-1})	y'_{initial} (km)	v_x^{initial} (m/s)	v_x^{final} (m/s)	v_y^{initial} (m/s)	v_y^{final} (m/s)	T (s)	s_x (km)	s_y (km)	a_0/g
Freefall (w. drogue)	0	0.072	0.112	10	250	12.6	1.0	118	40.6	3.4	4	1.5
Deploy 6 (2,2,2)	6	0.072	0.371	6	12.6	0	118	30.5	88.0	0	3	10.0
Deploy 18 (6,6,6)	24	0.072	0.716	3	0	0	30.5	13.7	203.0	0	3	2.7
Accumulated intervals									331.6	3.4	10	
Freefall (w/o drogue)	0	0	0.112	10	250	26.9	1.0	123	38.7	5.1	4	1.0
Deploy 6 (2,2,2)	6	0	0.3711	6	26.9	0	123	30.6	87.8	0.1	3	11.1
Deploy 18 (6,6,6)	24	0	0.716	3	0	0	30.5	13.7	203	0	3	2.7
Accumulated intervals									329.5	5.2	10	
Impact decel. length (m)												a_c/g
$L_c = 2$												4.8
$L_c = 3$												3.2

$v_x = 50$ m/s in 25.3 s while falling 2.01 km (from $h = 10$ km), attaining a vertical velocity $v_y = 117$ m/s. Deployment at that point of 24 G-11 parachutes would bring the plane to a terminal velocity $v_{t,y} = 13.7$ m/s, thereby subjecting passengers to an initial deceleration $a_0/g \sim 33.4$, which is beyond the assumed level of tolerance. In a safe recovery parachutes must be deployed sequentially and in a manner to keep the wings parallel to the ground (flat descent) so as to avoid unduly large initial decelerations.

An example of such a protocol, again obtained from numerical solution of eqs. (8), might unfold as follows. A B-747, cruising 250 m/s at 10 km becomes disabled; all engines fail or are shut off intentionally to effect the recovery. The drogue is deployed while the plane drops 4 km, which reduces v_x to 12.6 m/s and increases v_y to 118 m/s in about 40.6 s, at which time 6 vertical parachutes are deployed symmetrically in 3 groups of 2 along the fuselage. These decelerate the aircraft vertically to 30.5 m/s and horizontally to ~0 m/s, with peak deceleration $a_{\max} \sim 10g$, which decreases rapidly in time; the 5-second time-averaged deceleration is $a_{\text{av(5s)}} \sim 1.6g$. Then 18 more G-11 s are deployed symmetrically in 3 groups of 6, the total of 24 decelerating the aircraft ($a_{\max} \sim 2.7g$; $a_{\text{av(5s)}} = 0.3g$) to a terminal velocity $v_{t,y} = 1.7$ m/s, at which it falls the remaining distance to ground. The plane strikes the ground flat, compressing the cargo hold 2 m to produce an impact deceleration of less than $5g$.

Table 3 summarizes the kinematic details of the vertical descent from an initial altitude of 10 km both with and without use of a drogue. The two cases result in nearly the same maximum decelerations and a difference in cumulative horizontal displacement of less than 2 km. The preceding summary neglected the opening time of the parachute canopy, for which the mean delay Δt of a G-11 is about 5.3 s [15]. Taking account of this delay, however, by including time-dependent opening functions[1] in eqs. (8) did not perceptively change the numerical results of table 3 since $\Delta t \ll T$, the descent time at each deployment stage. Calculations were also performed for descents from lower initial altitudes. Space limitations preclude display of all results, but I note that the same recovery protocol, initiated at an altitude of only 4 km with deployments at 2.5 and 1.5 km also led to recovery with peak deployment and impact accelerations below $10g$.

Discussion and conclusions. – I have derived and examined an exact set of coupled equations, valid at high Reynolds number, for the freefall of an unpowered load through an atmosphere with the thermal characteristics of the Earth's troposphere. For general aviation aircraft $Re \geqslant\sim 10^6$. Numerical solutions for sequential, symmetric deployment of vertical and horizontal decelerators with drag parameters comparable to the largest commercially

[1]Unpublished calculations by the present author. The rate of areal increase of the canopy can be modeled by the equation $\mathrm{d}S/\mathrm{d}t = a + kS$, where a and k are constants, leading to the solution $S(t) = S_0(e^{kt} - 1)(e^{k\Delta t} - 1)^{-1}$ for $0 \leqslant t \leqslant \Delta t$ in which S_0 is the full canopy area and Δt is the delay interval. Inclusion of $S(t)$ with $\Delta t = 5.3$ s into eqs. (8) does not change the results of this paper for any real value of k.

available parachutes predict that such aircraft can be brought down in flat descent with mean (5 s interval) deployment and impact decelerations below $10g$.

Current barriers to such recovery are not aerodynamic, but, at most, material. The peak horizontal drag exerted by an air stream at 10 km altitude with relative velocity of 250 m/s on a 7.62 m radius drogue is ~ 3.7 MN, which amounts to a tension of 30.5 kN in each of the 120 suspension lines of diameter about 3.175 mm (0.125 inch), thereby requiring a tensile strength of about 3.9 GPa. Although a drogue may in fact be dispensable, peak drag on a G-11 vertical parachute corresponding to a relative vertical stream velocity of $\sim$123 m/s is ~ 3.5 MN, thereby requiring nearly the same tensile strength of 3.7 GPa. The tensile strength of the currently used type-III Nylon cord is about 309 MPa [16]. (Pressure constraints on the canopies are much less severe; peak drag overpressure on the drogue was $\sim$0.20 atm in the preceding analysis.)

There exist other materials, however, whose tensile strength is already within the range needed, and which may serve as precursors to suitable replacements for Nylon, such as a) Vectran (2.9–3.3 GPa), an aromatic polyester spun from a liquid-crystal polymer [17], b) Zylon (5.8 GPa), a thermoset liquid crystalline polybenzoxazole [18], and c) fiber glasses such as E-Glass (3.5 GPa) and S-Glass (4.7 GPa). Potentially new materials of extraordinary tensile strength may eventually be fabricated from allotropes of carbon with cylindrical nanostructure (C-nanotubes) which have the highest tensile strength of any known material (composites 2.3–14.2 GPa; single fibers 22.2 GPa) [19]. Successful implementation of the recovery protocols may also call for distributing the reaction force of the suspension lines over space or time to avoid structural damage at sites of attachment. This should be readily achievable by appropriate design of canopy shapes, controlled timing of canopy opening, and use of extensible materials.

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