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Using numerical methods to explore the space of solutions of a nonlinear partial differential equation

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Using numerical methods to explore the space of solutions of a nonlinear partial differential equation

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Contents

1 Standard Notations

Ω : bounded, open, connected set in $\mathbb{R}^n$

$\partial\Omega$: boundary of Ω

$C^r(\Omega)$: the space of r times continuously differentiable functions, $u : \Omega \to \mathbb{R}$

$O(h^p)$: If $f(h)$ and $g(h)$ are two functions of h, then we say that $f(h) = O(g(h))$ as $h \to 0$ if there is some constant C such that

$$\left| \frac{f(h)}{g(h)} \right| < C \quad \text{for all } h \text{ sufficiently small.}$$

$o(h^p)$: If $f(h)$ and $g(h)$ are two functions of h, then we say that $f(h) = o(g(h))$ as $h \to 0$ if

$$\left| \frac{f(h)}{g(h)} \right| \to 0 \quad \text{as } h \to 0.$$

$u(x)^+$: $\max\{0, u(x)\}$

$u(x)^-$: $-\min\{u(x), 0\}$

$\text{amp}(y)$: $\max\limits_{t\in[0,T]} y(t) - \min\limits_{t\in[0,T]} y(t)$ for $y : \mathbb{R} \to \mathbb{R}$ such that $t \mapsto y(t)$ with $y(t+T) = y(t)$.

$\text{amp}(u)$: $\max\limits_{x\in[0,\pi]} \left[\max\limits_{t\in[0,T]} u(x,t) - \min\limits_{t\in[0,T]} u(x,t) \right]$ for $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $(x,t) \mapsto u(x,t)$ with $u(x, t+T) = u(x,t)$.

u_h : Finite Difference approximation to u calculated on a grid with size h.

u_t : $\frac{\partial u}{\partial t}$

u_{xx} : $\frac{\partial^2 u}{\partial x^2}$

$\|u(x,t)\|_{H^1} = \sqrt{\|u\|^2_{L^2} + \|u_x\|^2_{L^2} + \|u_t\|^2_{L^2}}$

$\|u\|_{L^2} = \sqrt{\int_0^T \int_0^L |u(x,t)|^2 dx dt}$

2 Introduction

2.1 Beam Equation and Boundary Conditions

Beams are traditionally thought of as structural elements for buildings and engineering. However, any structures such as automotive automobile frames, aircraft components, machine frames, and other mechanical or structural systems that are designed to carry lateral loads can be analyzed in a similar fashion. The simplest partial differential equation modeling the vibrations of a one dimensional beam of length L is called the Euler Bernoulli beam equation and was introduced by Daniel Bernoulli and Euler in 1735,

$$u_{tt}(x,t) + Ku_{xxxx}(x,t) = F(x,t), \quad (x,t) \in (0,L) \times (0,\infty) \tag{1}$$

where $u(x,t)$ is the vertical position at time t and length x along the beam and $F(x,t)$ is force acting on the beam. This equation involves the fourth x-derivative of u, instead of the second derivative that occurs in the wave equation (34). The boundary conditions on the equation model supports, but they can also model point loads, distributed loads and moments. The support or displacement boundary conditions are used to fix values of displacement (u) and rotations (u_x) on the boundary. At $x = 0$ endpoint the beam can either be clamped ($u(0,t) = u_x(0,t) = 0$), hinged ($u(0,t) = u_{xx}(0,t) = 0$) or free ($u_{xx}(0,t) = u_{xxx}(0,t) = 0$). Similarly, the beam can be clamped, hinged or free at $x = L$ endpoint. The figures below illustrate the various ways in which beams can be supported.

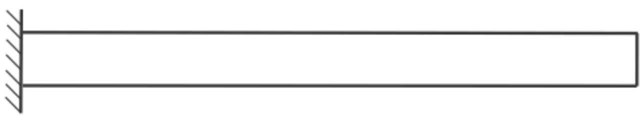


Figure 1: Cantilever beam

A cantilever beam is clamped at one end and free at the other end as in figure (1). A diving board is an example of a beam supported in this way. There is no displacement or rotation at the clamped end.

$$u(0,t) = u_x(0,t) = 0$$
$$u_{xx}(L,t) = u_{xxx}(L,t) = 0$$

Figure 2: Simply Supported beam

A simply supported beam is hinged at both ends as in figure (2). These are free to rotate about the fixed end points. A suspension bridge is an example of a beam supported this way.

$$u(0,t) = u_{xx}(0,t) = 0$$
$$u(L,t) = u_{xx}(L,t) = 0$$

Figure 3: Fixed beam

A fixed beam is clamped at both of its ends and are fixed in place as in figure (3). Beams supported this way are especially common in structures.

$$u(0,t) = u_x(0,t) = 0$$
$$u(L,t) = u_x(L,t) = 0$$

Figure 4: Overhanging beam

An overhanging beam is a beam extending beyond its support on one end. One of the example is a beam with one end hinged and another end free as in figure (4).

$$u(0,t) = u_{xx}(0,t) = 0$$
$$u_{xx}(L,t) = u_{xxx}(L,t) = 0$$

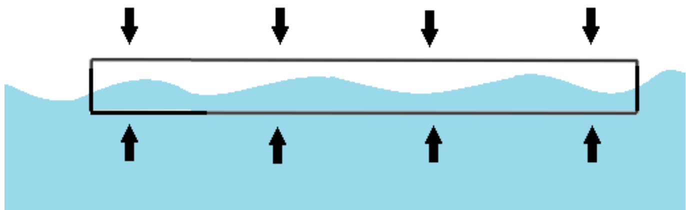


Figure 5: Free floating beam

Another example of a beam is one with both ends free. Since it is not supported at either end, it could be interpreted as a beam floating on the water. It is free to change displacement and rotation at the endpoints. The boundary conditions for the free floating beam can be written as:

$$u_{xx}(0,t) = u_{xxx}(0,t) = 0$$
$$u_{xx}(L,t) = u_{xxx}(L,t) = 0$$

2.2 The partial differential equation

We investigate the periodic solutions of non-linearly supported periodically forced beams with some of the boundary conditions discussed above. Non-linearly suspended beams can be used as models for suspension bridges [1] and ships at sea [2]. Another non-linearly supported system is a mass spring system with a cable providing additional support [3]. The cable resists expansion but not compression which introduces a non-linearity to the system.

We explore equations of the form,

$$u_{tt} + cu_{xxxx} + du_t + bu^+ = g + \lambda \sin(\mu t), \quad (x,t) \in (0,L) \times (0,\infty). \tag{2}$$

The partial differential equation (2) was developed in [2] as a model of a long ship at sea where $u(x,t)$ represent the submerged depth of the beam at position x and time t. The constant c is the beam equation constant, d represents damping coefficient/air resistance, b measures the "stiffness" of cable in one-sided Hooke's law, g is the gravitational constant ($9.81ms^{-2}$) and $\lambda \geq 0$ and $\mu > 0$. The term $u^+ = \max(u,0)$ models the fact that the buoyant force which keeps the beam afloat is proportional with constant b, to the submerged depth, unless the beam is out of the water ($u < 0$) in which case the buoyant force is 0. The u^+ term can also model the fact that a beam suspended by cables (which act like "nonlinear" springs) resists expansion but not compression (for example, a rubber band).

We investigate how the number, amplitude and stability of periodic responses depends on the forcing amplitude λ and forcing frequency μ by constructing bifurcation diagrams for different boundary conditions for periodic forcing functions $f(x,t) = f(x,t;\lambda,\mu)$. Understanding steady state solutions (equilibrium and steady-state responses to periodic forcing) is useful to understand the dynamics. Since period of forcing term is $T = \frac{2\pi}{\mu}$, we look for solutions that are T-periodic in time, by imposing a further condition

$$u(x,t+T) = u(x,t) \text{ for } (x,t) \text{ in } (0,L) \times (0,\infty)$$

in addition to the boundary conditions that the equation that we are solving for must satisfy. In this thesis, we will focus on $\mu = 4$, but the same methods could be applied to other values of μ. The values of the parameters comes from [2] and are chosen to represent a "relatively flexible beam."

We start by developing necessary numerical methods with simpler examples in section 4 and use those methods to solve equation (2) with various boundary conditions in section 5.

3 The Ordinary Differential Equation

We will start by showing how the PDE in (2) can be *loosely connected* to an ODE model of a nonlinear oscillator. Assume $u(x,t) = y(t)\sin(\frac{\pi x}{L})$, a standing wave with single node

$$\begin{aligned}
u_{xxxx}(x,t) &= \left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi x}{L}\right) y(t) \\
u_{tt} &= y''(t)\sin\left(\frac{\pi x}{L}\right) \\
u_t &= y'(t)\sin\left(\frac{\pi x}{L}\right) \\
u^+ &= \left(\sin\left(\frac{\pi x}{L}\right) y(t)\right)^+ = \sin\left(\frac{\pi x}{L}\right)(y(t))^+
\end{aligned}$$

Substituting these values in (2), we get

$$\begin{aligned}
y''(t)\sin\left(\frac{\pi x}{L}\right) + c\left(\frac{\pi}{L}\right)^4 \sin\left(\frac{\pi x}{L}\right) y(t) + dy'(t)\sin\left(\frac{\pi x}{L}\right) + b\sin\left(\frac{\pi x}{L}\right).(y(t))^+ &= g + \lambda\sin(\mu t) \\
\sin\left(\frac{\pi x}{L}\right)\left(y''(t) + c\left(\frac{\pi}{L}\right)^4 y(t) + dy'(t) + by(t)^+\right) &= g + \lambda\sin(\mu t)
\end{aligned}$$

Using $g = \sin(\frac{\pi x}{L})\hat{g}$ and $\lambda = \sin(\frac{\pi x}{L})$
$hat\lambda$ on the right hand side, we can rewrite the equation as,

$$y'' + \delta y' + ay^+ - by^- = \hat{g} + \hat{\lambda}\sin(\mu t). \tag{3}$$

If k_s is the spring stiffness and k_c is the cable stiffness, then $a = (k_s + k_c)$, $b = k_s$.

Equation (3) can be interpreted as a mass attached to a vertical spring with a cable providing additional support [3]. While the spring causes a restoring force in both the upward and downward directions, the cable only resists expansion. Let $y(t)$ denote the downward displacement of the mass at time t, where $y = 0$ denotes the position before elongation of the spring by the addition of the mass. There are three main forces acting on the mass: a “one-sided” or nonlinear restoring force from the cable, a linear restoring

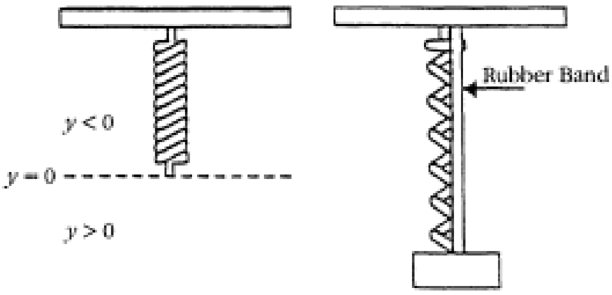


Figure 6

force from the spring and gravity.

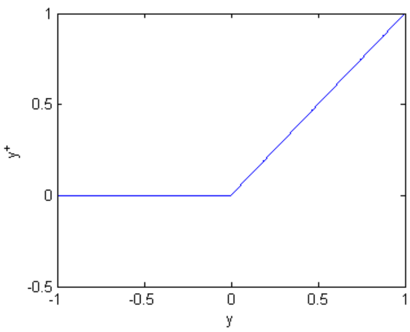


Figure 7: Plot of y vs y^+

We will develop and explain the methods that search for periodic responses to (3), namely solutions satisfying the boundary conditions given by $y(0) = y(T)$, $y'(0) = y'(T)$ to look for periodic solutions for a period T as we vary λ. Note, if $\lambda = 0$, there is a natural equilibrium $y = g/a$ and the particle obeys the linear equation

$$y'' + \delta y' + ay = g.$$

We will use this natural equilibrium as our initial value with $\lambda = 0$ for the parameter continuation algorithm described in the next section. The behavior of this system has been widely studied allowing us to check that our methods are valid and will also help to gain intuition regarding the behaviour of solutions of (2). We will look at the plot of the amplitude of the solution versus a parameter of the equation, in our case the amplitude of the forcing term λ. The amplitude of the oscillation is half the difference between the maximum and minimum of the oscillation over a time period.

4 Numerical Techniques

In this section, we describe various numerical methods that we use throughout.

4.1 Numerical Differentiation

We want to find a function (or some discrete approximation to this function) that satisfies a given relationship between various of its derivatives on some given region along with some boundary conditions.

Definition 1. *Given a function $u(x)$, an order k and a point $\tilde{x}$, a finite difference approximation to $\frac{\partial^k u}{\partial x^k}(\tilde{x})$ is linear combination of values of u at points near $\tilde{x}$.*

Definition 2. *The truncation error of a finite difference method is the difference between the approximation and exact analytical solution. We will denote the approximation by u_h when the order k and $\tilde{x}$ is clear from the context, in this case, error $= \frac{\partial^k u}{\partial x^k}(\tilde{x}) - u_h$.*

Definition 3. *An order p approximation method to $\frac{\partial^k u}{\partial x^k}(\tilde{x})$ is a method to obtain u_h such that error is $O(h^p)$, where h is uniform spacing.*

4.1.1 First Derivative

Consider a function of one variable $u(x)$. We assume that u is continuously differentiable over an interval. The derivative of u at a point $\tilde{x}$, denoted $u'(\tilde{x})$, if it exists can be approximated using a finite difference approximation based on values of u at a finite number of points near $\tilde{x}$ to obtain

$$D_+u(\tilde{x}) = \frac{u(\tilde{x}+h) - u(\tilde{x})}{h}. \tag{4}$$

for some small value of h. The limiting value of this expression equals the derivative of $u(x)$ at $\tilde{x}$ as $h \to 0$ from the standard definition of derivative. Equation (4) is called forward difference approximation of $u'(\tilde{x})$. We can derive this approximation using a Taylor expansion.

Theorem 1 (Taylors theorem)**.** *Let f be $(k+1)$ times continuously differentiable in a neighborhood of $\tilde{x}$. Then there exists a number c between x and $\tilde{x}$ such that*

$$\begin{aligned} f(x) = f(\tilde{x}) + f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \frac{f'''(\tilde{x})}{3!}(x-\tilde{x})^3 + \cdots \\ + \frac{f^{(k)}(\tilde{x})}{k!}(x-\tilde{x})^k + \frac{f^{(k+1)}(c)}{(k+1)!}(x-\tilde{x})^{k+1}. \end{aligned} \tag{5}$$

The terms up to degree k in $(x-\tilde{x})$, is called the degree k Taylor polynomial for f centered at $\tilde{x}$. The last term is called the Taylor remainder.

Assuming f is twice continuously differentiable in the interval where it is defined, we can write the Taylor expansion of $f(\tilde{x}+h)$ as

$$f(\tilde{x}+h) = f(\tilde{x}) + hf'(\tilde{x}) + \frac{h^2}{2}f''(\tilde{x}) + O(h^3) \tag{6}$$

where $O(h^3)$ denotes that the remainders in the Taylor Theorem is of order 3. Rearranging (6) gives,

$$\begin{aligned} f'(\tilde{x}) &= \frac{f(\tilde{x}+h) - f(\tilde{x})}{h} - \frac{h}{2}f''(\tilde{x}) + \frac{O(h^3)}{h} \\ &= \frac{f(\tilde{x}+h) - f(\tilde{x})}{h} + O(h) \end{aligned} \tag{7}$$

Using (7) to approximate the derivative, we obtain the truncation error

$$f'(\tilde{x}) - D_+u(\tilde{x}) = O(h).$$

This also shows that (4) is a first order method for approximating the first derivative.

A better approximation can be obtained by using two sided difference known as centered difference formula,

$$D_0u(\tilde{x}) = \frac{u(\tilde{x}+h) - u(\tilde{x}-h)}{2h}. \tag{8}$$

Again, assuming that $f \in C^2$, we can Taylor expand $f(\tilde{x}+h)$ and $f(\tilde{x}-h)$,

$$f(\tilde{x}+h) = f(\tilde{x}) + hf'(\tilde{x}) + \frac{h^2}{2}f''(\tilde{x}) + O(h^3) \tag{9}$$

$$f(\tilde{x}-h) = f(\tilde{x}) - hf'(\tilde{x}) + \frac{h^2}{2}f''(\tilde{x}) - O(h^3) \tag{10}$$

Subtracting (10) from (9) and rearranging gives,

$$\begin{aligned} f'(\tilde{x}) &= \frac{f(\tilde{x}+h) - f(\tilde{x}-h)}{2h} - \frac{O(h^3)}{h} \\ &= \frac{f(\tilde{x}+h) - f(\tilde{x}-h)}{2h} - O(h^2) \end{aligned} \tag{11}$$

From (11), we see that the (8) is second order accurate approximation. Thus, (8) gives better approximation than the one-sided approximation (4).

Similarly, we can develop a third order accurate approximation, (12) gives third order accurate approximation

$$D_3 u(\tilde{x}) = \frac{1}{6h}[2u(\tilde{x}+h) + 3u(\tilde{x}) - 6u(\tilde{x}-h) + u(\tilde{x}-2h)]. \tag{12}$$

A method for producing u_h is said to be convergent of order p if

$$\begin{aligned} error &= O(h^p) \quad \text{ as } h \to 0 \\ |u - u_h| &= Ch^p \\ \log(|u - u_h|) &= \log(Ch^p) \\ \log(|u - u_h|) &= k + p\log(h) \end{aligned}$$

The slope of loglog plot gives the order of the approximation. Figure (8) shows errors using approximations $D_+u(\tilde{x})$, $D_0u(\tilde{x})$ and $D_3u(\tilde{x})$ plotted against grid size h on a loglog plot.

In general, we can derive finite difference approximations of any order to $u'(\tilde{x})$ based on sufficiently many nearby points using Taylor series and method of undetermined coefficients. It is important to note that by Taylor expanding a function we are making an implicit assumption that it is sufficiently smooth. This method can be generalized to approximate $\frac{d^k u}{dx^k}(\tilde{x})$ up to p^{th} order. Suppose we want approximation to $u'(\tilde{x})$ based on $u(\tilde{x}), u(\tilde{x}-h)$, and $u(\tilde{x}-2h)$ of the form

$$D_2 u(\tilde{x}) = au(\tilde{x}) + bu(\tilde{x}-h) + cu(\tilde{x}-2h) \tag{13}$$

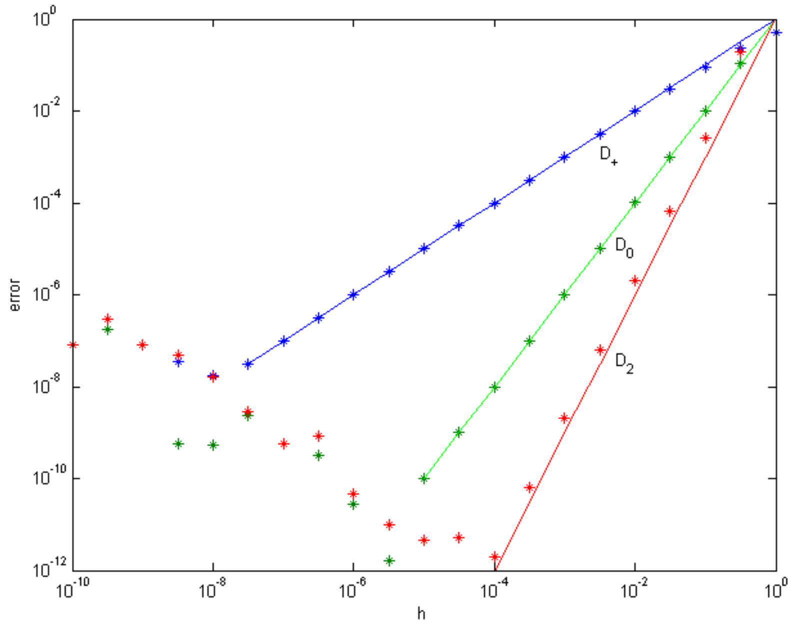


Figure 8: Error in approximation of derivative of $f(x) = e^x$ at $x = 1$ using three different approximations: forward difference approximation D_+, centered difference approximation D_0 and third order approximation D_3. Error is calculated using $f' = e^x$ and plotted against grid size h on a loglog scale. The method breaks when h is very small due to the rounding error caused by machine error.

We will use Taylor expansions of $u(\tilde{x}), u(\tilde{x}-h)$, and $u(\tilde{x}-2h)$ at $\tilde{x}$,

$$\begin{aligned}
u(x) =& u(\tilde{x}) + u'(\tilde{x})(x-\tilde{x}) + \frac{(x-\tilde{x})^2}{2!}u''(\tilde{x}) + \frac{(x-\tilde{x})^3}{3!}u'''(\tilde{x}) + O(x-\tilde{x})^4 \\
u(\tilde{x}-h) =& u(\tilde{x}) - hu'(\tilde{x}) + \frac{h^2}{2!}u''(\tilde{x}) - \frac{h^3}{3!}u'''(\tilde{x}) + O(h^4) \\
u(\tilde{x}-2h) =& u(\tilde{x}) - 2hu'(\tilde{x}) + \frac{4h^2}{2!}u''(\tilde{x}) - \frac{8h^3}{3!}u'''(\tilde{x}) + O(h^4)
\end{aligned} \tag{14}$$

Collecting the terms, we get

$$D_2u(\tilde{x}) = (a+b+c)u(\tilde{x}) - (b+2c)hu'(\tilde{x}) + \frac{1}{2}(b+4c)h^2u''(\tilde{x}) - \frac{1}{6}(b+8c)h^3u'''(\tilde{x}) + \cdots \tag{15}$$

For this to agree with $u'(\tilde{x})$ up to second order, we need

$$\begin{aligned} a + b + c &= 0 \\ b + 2c &= -1/h \\ b + 4c &= 0 \end{aligned} \tag{16}$$

Solving the linear system 16 gives

$$a = \frac{1}{h^2}, \quad b = -\frac{2}{h^2}, \quad c = \frac{1}{h^2}$$

so the formula for second order approximation for $u'(\tilde{x})$ is

$$D_2 u(\tilde{x}) = \frac{1}{h^2}\big[u(\tilde{x}) - 2u(\tilde{x} - h) + u(\tilde{x} - 2h)\big] \tag{17}$$

As this is second order method, we expect the error of this approximation to be $O(h^2)$. To see this we compute the error as,

$$\begin{aligned} D_2 u(\tilde{x}) - u'(\tilde{x}) &= -\frac{1}{6}(b + 8c)h^3 u'''(\tilde{x}) + \cdots \\ &= \frac{1}{12}h^2 u'''(\tilde{x}) + \cdots \\ &= O(h^2) \end{aligned} \tag{18}$$

4.1.2 Second Derivative

Approximations to the second derivative $u''(x)$ can be obtained in an analogous manner. Suppose we want standard second order centered approximation to $u''(\tilde{x})$ based on $u(\tilde{x}-h)$, $u(\tilde{x})$, and $u(\tilde{x} + h)$ of the form

$$D^2 u(\tilde{x}) = au(\tilde{x} - h) + bu(\tilde{x}) + cu(\tilde{x} + h) \tag{19}$$

The Taylor series expansions of $u(\tilde{x}), u(\tilde{x}-h)$, and $u(\tilde{x}+h)$ at $\tilde{x}$ are

$$
\begin{aligned}
u(x) &= u(\tilde{x}) + u'(\tilde{x})(x-\tilde{x}) + \frac{(x-\tilde{x})^2}{2!}u''(\tilde{x}) + \frac{(x-\tilde{x})^3}{3!}u'''(\tilde{x}) + O(x-\tilde{x})^4 \\
u(\tilde{x}-h) &= u(\tilde{x}) - hu'(\tilde{x}) + \frac{h^2}{2!}u''(\tilde{x}) - \frac{h^3}{3!}u'''(\tilde{x}) + O(h^4) \\
u(\tilde{x}+h) &= u(\tilde{x}) + hu'(\tilde{x}) + \frac{h^2}{2!}u''(\tilde{x}) + \frac{h^3}{3!}u'''(\tilde{x}) + O(h^4)
\end{aligned}
\tag{20}
$$

Collecting the terms, we get

$$
D^2u(\tilde{x}) = (a+b+c)u(\tilde{x}) + (a-c)hu'(\tilde{x}) + \frac{1}{2}(a+c)h^2u''(\tilde{x}) - \frac{1}{6}(a-c)h^3u'''(\tilde{x}) + \cdots \tag{21}
$$

For this to agree with $u''(\tilde{x})$ up to fourth order, we need

$$
\begin{aligned}
a+b+c &= 0 \\
a-c &= 0 \\
a+c &= \frac{2}{h^2}
\end{aligned}
\tag{22}
$$

Solving the linear system (22) gives

$$
b = -\frac{2}{h^2}, \quad a = c = \frac{1}{h^2}
$$

so the formula for second order approximation for $u''(\tilde{x})$ is

$$
D^2u(\tilde{x}) = \frac{1}{h^2}\left[u(\tilde{x}-h) - 2u(\tilde{x}) + u(\tilde{x}+h)\right] \tag{23}
$$

We calculate the truncation error in this approximation as

$$
\begin{aligned}
D^2u(\tilde{x}) - u''(\tilde{x}) &= \frac{1}{24}(a+c)h^4u^{(4)}(\tilde{x}) + \cdots \\
&= \frac{h^2}{12}u^{(4)}(\tilde{x}) + \cdots \\
&= O(h^2)
\end{aligned}
\tag{24}
$$

This shows that the method is second order.

4.2 Initial value Solvers for first order Systems

In this section, we introduce methods to solve initial value problems (IVP) for first order systems in time. We start by discretizing in time for fixed time step k, such that $t^n = nk$ will use superscript for the time step index and subscript for the spatial indices. Given values at the initial time ($t = 0$), $U^0 = [u_1^0, u_2^0, ..., u_m^0]$, we will use initial value solvers to solve for $U^1, U^2, \cdots$ satisfying $U^n \approx U(t_n)$. Given $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, consider IVP of the form,

$$\begin{aligned} \frac{d}{dt}U &= f(t, U) \\ U(0) &= U^0 \end{aligned} \tag{25}$$

The simplest method to solve (25) is *Euler's Method* (also called Forward Euler). It is based on replacing $u'(t^n)$ with $D_+U^n = (U^{n+1} - U^n)/k$ from (4). This gives,

$$\frac{U^{n+1} - U^n}{k} = f(t^n, U^n)$$

$$U^{n+1} = U^n + kf(t^n, U^n). \tag{26}$$

Thus, from the initial data $U^0 = U(t^0)$ we can compute $U^1 = U(t^1)$, and so on.

The *backward Euler* method is similar but is based on replacing $u'(t^{n+1})$ with D_-U^{n+1} to get,

$$\frac{U^{n+1} - U^n}{k} = f(t^{n+1}, U^{n+1})$$

$$U^{n+1} = U^n + kf(t^{n+1}, U^{n+1}). \tag{27}$$

In the backward Euler method, (27) is an equation that must be solved for U^{n+1}. If $f(t, U)$ is linear, we can write $f(t, U) = AU$ for some $m \times m$ matrix A and U^{n+1} can be obtained explicitly by solving the linear system,

$$U^{n+1} = (I - kA)^{-1}U^n$$

where I is $m \times m$ identity matrix. If $f(t, U)$ is a nonlinear function, we can write this as

$$U^{n+1} - kf(U^{n+1}) - U^n = 0 \tag{28}$$

and notice that U^{n+1} is a root of

$$g(u) = u - kf(u) - U^n, \tag{29}$$

Note that U^n is known from the previous time step or as the initial condition when $n = 0$. Thus, solving for $g(u) = 0$ gives values of U^{n+1} that satisfies (27). The root of equation (29) can be approximated using some iterative methods such as *Newton's Method.*

Theorem 2 (Newtons Method)**.** *Given a sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a suitable initial guess x_0, the root of the function $f(x)$ can be approximated using the iterative algorithm*

$$x_{n+1} = x_n - [\frac{\partial f}{\partial x} x_n]^{-1} f(x_n)$$

where the Jacobian $\frac{\partial f}{\partial x}$ exists and is invertible.

Since the backward Euler method gives an equation that must be solved for U^{n+1}, it is called an implicit method, whereas the forward Euler method is an explicit method.

Another implicit method is the trapeziodal method (also called *Crank-Nicolson Method*), obtained by averaging the two Euler methods:

$$\frac{U^{n+1} - U^n}{k} = \frac{1}{2}\left(f(U^n) + f(U^{n+1})\right) \tag{30}$$

The approximation in (30) is second order accurate in k, whereas the Euler methods are only first order accurate.

4.3 Solving an ordinary differential equation using finite difference method

Finding the exact (analytical) solution of differential equations is usually very difficult even if they are linear. Most differential equations do not have analytical solutions and thus require a numerical procedure to find an approximate solution. A finite difference method approximates the derivatives with finite differences at discrete values of the independent variable. This results in a system of equations that can be solved in place of the differential equation to obtain an approximation to the solution.

We will use difference formulas from previous sections to solve differential equations on an interval $[a, b]$. In general, to solve a differential equation in an interval we will start by selecting the number of interior grid points N at which we want to approximate our solution and set $h = \frac{b-a}{N+1}$. Set $x_i = a + ih$ for $i = 0, 1, \cdots, N + 1$ so that $x_0 = a$ and $x_{N+1} = b$ and solve our equation at the discrete points x_i. Consider a single variable Poisson problem with Dirichlet boundary condition given as

$$u''(x) = f(x), \quad x \in (a, b)$$
$$u(a) = \alpha, \quad u(b) = \beta$$

Using our centered finite difference approximation for u'' to solve the equation for each x_i, gives

$$\frac{1}{h^2}\left(u(x_{i-1}) - 2u(x_i) + u(x_{i+1})\right) = f(x_i), \quad i = 1, 2 \cdots 1, 2, \cdots, N$$

Writing the individual equation for each x_i, gives

$$\begin{aligned} \frac{1}{h^2}(u(x_0) - 2u(x_1) + u(x_2)) &= f(x_1) \\ \frac{1}{h^2}(u(x_1) - 2u(x_2) + u(x_3)) &= f(x_2) \\ &\vdots \\ \frac{1}{h^2}(u(x_{N-1}) - 2u(x_N) + u(x_{N+1})) &= f(x_N) \end{aligned} \tag{31}$$

Denoting $u_i = u(x_i)$ and letting $\mathcal{U} = [u_1, u_2, \cdots, u_N]^T$ and $\mathcal{F} = [f(x_1), f(x_2), \cdots, f(x_N)]^T$, and using $u(x_0) = u(a) = \alpha$ and $u(x_{N+1}) = u(b) = \beta$ the system of equations can be written as

$$A\mathcal{U} = \mathcal{F} \tag{32}$$

$$\text{where } A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \text{ and } \mathcal{F} = \begin{bmatrix} f(x_1) - \frac{1}{h^2}\alpha \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) - \frac{1}{h^2}\beta \end{bmatrix}$$

This tridiagonal system is non-singular and can be solved for the interior grid points, $\mathcal{U}$. The first and the last equations in the system of equations (31) involve the boundary points $u(x_0)$ and $u(x_{N+1})$ whose value is known from the boundary condition. Thus, it is subtracted in the right hand side in the first and last row of $\mathcal{F}$.

4.4 Solving a partial differential equation using finite difference method

We use the methods developed to solve a second order partial differential equation, Laplace equation. The Laplace equation and Poisson equation are the simplest examples of elliptic partial differential equations. If $u \in C^2$, the Laplace operator or "Laplacian" is defined as

$$\Delta u = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \text{div}(\text{grad } u).$$

If the right-hand side is specified as some function f, such as

$$\Delta u = f$$

then it is called Poisson's equation. The Poisson equation in two independent variables has the form

$$\Delta u = u_{xx} + u_{yy} = f(x, y) \tag{33}$$

We can use similar method as previous section to solve (33). However, now we will need to discretize in two different dimensions x and y. We replace the x and y derivatives with centered finite differences, which gives

$$\frac{1}{(h_x)^2}(u_{i-1,j} - 2u_{ij} + u_{i+1,j}) + \frac{1}{(h_y)^2}(u_{i,j-1} - 2u_{ij} + u_{i,j+1}) = f_{ij}$$

here $u_{i,j} = u(x_i, y_j)$ and $f_{ij} = f(x_i, y_j)$. Consider the simplified case where $h_x = h_y = h$, then we get

$$\frac{1}{h^2}(u_{i-1,j} + u_{i+1,j} - 4u_{ij} + u_{i,j-1} + u_{i,j+1}) = f_{ij}.$$

If we write the vector of unknowns as $U = \begin{bmatrix} u^{[1]} \\ u^{[2]} \\ \vdots \\ u^{[N]} \end{bmatrix}$ where $u^{[j]} = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{Nj} \end{bmatrix}$, we can write (33) as

$$\mathcal{A}U = F$$

where $\mathcal{A}$ has the form

$$\mathcal{A} = \frac{1}{h^2}\begin{bmatrix} T & I & & & & \\ I & T & I & & & \\ & I & T & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & T & I \\ & & & & I & T \end{bmatrix}$$

which is an $N \times N$ block tridiagonal matrix in which each block T and I are also $N \times N$ matrix with

$$T = \begin{bmatrix} -4 & 1 & & & & \\ 1 & -4 & 1 & & & \\ & 1 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 1 \\ & & & & 1 & -4 \end{bmatrix}$$

and $F = \begin{bmatrix} f^{[1]} \\ f^{[2]} \\ \vdots \\ f^{[N]} \end{bmatrix}$ where $u^{[j]} = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{Nj} \end{bmatrix}$.

Solving the Heat Equation

As a specific example, we solve a differential equations with both spatial and time variables. Consider the flow of heat in some homogeneous heat-conducting material, subject to some external heat source along its length and some boundary conditions at each end. If we assume that the material properties, the initial temperature distribution, and the source vary only with x, the distance along the length, and not across any cross section, then we expect the temperature distribution at any time to vary only with x and we can model this with a differential equation in one space dimension. Since the solution might vary with time, we let $u(x,t)$ denote the temperature at point x and at time t. The solution is then governed by the heat equation

$$u_t = \kappa u_{xx} + f(x,t) \tag{34}$$

where κ is the coefficient of heat conduction, and $f(x,t)$ is an external heat source.

Along with the equation, we need initial conditions (35) and boundary conditions (36-37).

$$u(x,0) = u^0(x) \tag{35}$$

We consider the boundary conditions called Dirichlet boundary conditions (36), where the temperature at each end is specified as opposed to Neumann boundary condition (37) which is a condition on the derivative of u rather than on u itself. The Neumann boundary conditions translate as one end or both ends might be insulated, in which case there is zero heat flux at that end and so $u_x = 0$ at the point.

$$u(a,t) = \alpha(t), \quad u(b,t) = \beta(t) \tag{36}$$

$$u_x(a,t) = \alpha t, \quad u_x(b,t) = \beta(t) \tag{37}$$

Consider the heat equation (34) with no external heat source and Dirichlet boundary conditions given as,

$$\begin{aligned} u_t = \kappa u_{xx}, \quad x \in (a,b) \\ u(x,0) = g(x) \\ u(a,t) = u(b,t) = 0 \end{aligned}$$

Using second order central difference approximation for u_{xx} and Euler's method (26) to march forward in time, we can rewrite (34) as

$$u_t(x_i, t^k) = \kappa u_{xx}(x_i, t^k)$$

$$\frac{u_i^{k+1} - u_i^k}{k} = \kappa\left(\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}\right)$$

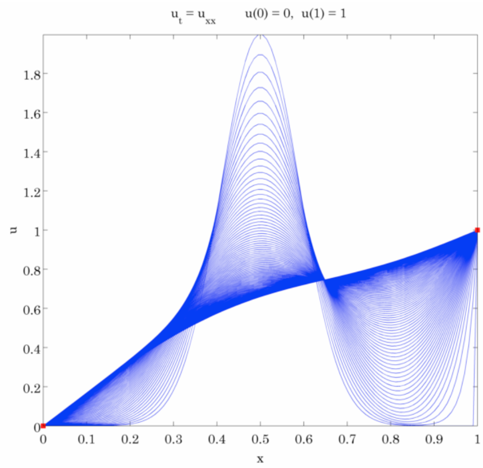
$$u_i^{k+1} = u_i^k + \kappa k\left(\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}\right) \tag{38}$$

Letting $U = [u_1, u_2, \cdots, u_N]^T$ and letting A be the $n \times n$ matrix in Equation 32 gives,

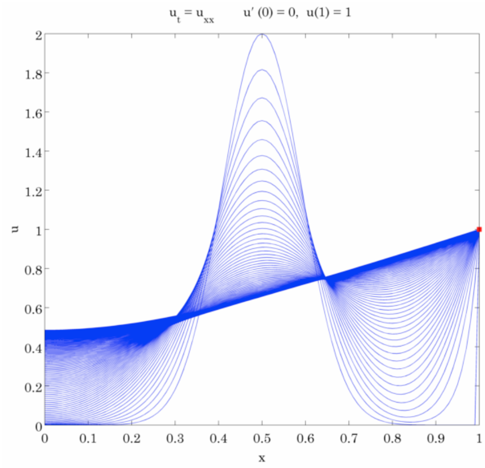
$$\begin{aligned} U^{k+1} =& U^k + \kappa k A U^k \\ =& (I + \kappa k A) U^k \end{aligned} \tag{39}$$

Thus, we can march forward in time.

In figure (**??**) the initial heat distribution in the center diffuses and approaches steady state solution $y = x$ and $y = 1$ for the Dirichlet (left) and Neumann boundary conditions (right) respectively.



(a) Solution of heat equation with Dirichlet boundary conditions.



(b) Solution of heat equation with Neumann boundary condition at one end and Dirichlet boundary condition at other end.

4.5 Numerical Continuation

Definition 4. *Numerical continuation is a method of numerically computing approximate solutions of parametrized non-linear equations of the form*

$$F(\boldsymbol{u}, \lambda) = 0,$$

where the parameter λ is usually a real scalar and u is a vector in $\mathbb{R}^n$ representing our discretized versions of solutions.

A numerical continuation algorithm takes as its input a system of parametrized nonlinear equations and an initial solution and produces a set of points on the solution component. The solution component is comprised of what are called regular points and singular points.

Definition 5. *A solution component $\Gamma(\boldsymbol{u}_0, \lambda_0)$ of the nonlinear system F is a set of points $(\boldsymbol{u}, \lambda)$ which satisfy $F(\boldsymbol{u}, \lambda) = 0$ and are connected to the initial solution $(\boldsymbol{u}_0, \lambda_0)$ by a path of solutions $(\boldsymbol{u}(s), \lambda(s))$.*

We classify the points as regular, singular or turning points using the Implicit Function Theorem.

Theorem 3. *If $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth function with $F(\boldsymbol{u}_0, \lambda_0) = 0$ where $\boldsymbol{u}_0 \in \mathbb{R}^n$,*

$\lambda_0 \in \mathbb{R}$ *and* $\det \frac{\partial F}{\partial \mathbf{u}}(u_0, \lambda_0) \neq 0$ *and smooth. Then there exists a unique smooth function* $\mathbf{u}(\lambda)$ *such that* $F(\mathbf{u}(\lambda), \lambda) = \mathbf{0}$ *for all* λ *near* λ_0 *and* $\mathbf{u}(\lambda_\mathbf{0}) = \mathbf{u_0}$.

The Jacobian of the system is the $N \times (N+1)$ matrix given as

$$DF = \begin{bmatrix} \frac{\partial F}{\partial \mathbf{u}} & \frac{\partial F}{\partial \lambda} \end{bmatrix} \tag{40}$$

Definition 6. *A regular point of* F *is a point* $(\mathbf{u}, \lambda)$ *at which the Jacobian of* F*,* DF *is full rank. Near a regular point the solution component is an isolated curve passing through the regular point.*

Definition 7. *A singular point of* F *is a point* $(\mathbf{u}, \lambda)$ *at which the Jacobian of* F*,* DF *is not full rank. Near a singular point the solution component may not be an isolated curve passing through the point.*

Definition 8. *A turning point of* F *in* λ *is a point* $(\mathbf{u}, \lambda)$ *at which* $\det \frac{\partial F}{\partial \mathbf{u}} = 0$ *but the Jacobian of* F*,* DF *has full rank.*

In general solution components Γ are branching curves or closed loops. Unfortunately, the assumptions of Implicit Function Theorem do not apply as we approach singular points or points where new branches of the solution components are created. Thus, we will need more sophisticated techniques to trace solution curves through branch points something that we are not concerning ourselves with.

4.5.1 Natural parameter Continuation

Most numerical methods for solving nonlinear systems of equations are iterative methods. For a particular parameter value a mapping is repeatedly applied to an initial guess. If the method converges, and is consistent, then in the limit the iteration approaches a solution of the nonlinear system.

Natural parameter continuation is a an adaptation of the iterative solver to a parametrized problem. The solution at one value of λ is used as the initial guess for the solution at $\lambda + \Delta\lambda$. With $\Delta\lambda$ sufficiently small the iteration applied to the initial guess should converge.

Consider as an example,

$$F(y, \lambda) = \lambda^2 + y^2 - 1 \tag{41}$$

It is known that equation (41) traces a circle as λ varies from -1 to 1 and one solution of the equation is $(0, 1)$, i.e, $\lambda_0 = 0$ and $y_0 = 1$. Using this known solution as an initial guess, we can find solutions to the equation for different values of the parameter λ using iterative methods. Increment λ by small amount $\Delta\lambda$ and use the solution, y_0 as the initial guess for the solution y at $\lambda + \Delta\lambda$. The iterative method (such as Newtons Method) refines the initial guess to a solution at $\lambda + \Delta\lambda$.

Equation (41) is smooth function with $F(y_0, \lambda_0) = 0$ and the derivative $\frac{\partial F}{\partial y} = 2y \neq 0$ when $y \neq 0$. Thus, the implicit function theorem tells us that there's a unique smooth function $y(\lambda)$ such that $F(y(\lambda), \lambda) = 0$ for all λ near λ_0. However, at $y = 0$,

$$\frac{\partial F}{\partial y} = 2y = 0.$$

Thus, the assumptions of the implicit function theorem fails ($\frac{\partial F}{\partial y} = 0$), thus the natural parameter continuation method fails at the turning point $\lambda = 1$, Figure 9. Thus, we will develop a more sophisticated continuation algorithm known as pseudo arc-length continuation.

4.5.2 Pseudo arc-length continuation

The pseudo arc-length method is based on the observation that arc-length offers more natural parameterization of a curve. Pseudo arc-length is an approximation of the arc-length in the tangent space of the curve. The resulting modified natural continuation method makes a step in arc-length Δs rather than λ. Then we use secant predictor to obtain a initial guess for the iterative solver (Newtons Method solver) to refine the solution.

An additional equation of pseudo arclength normalization (42) is introduced that is dependent on initial conditions $\mathbf{u_0}$, λ_0 and the arc length parameter, s. If $(\mathbf{u_0}, \lambda_\mathbf{0})$ is a solution of F, we parametrize $(\mathbf{u_0}, \lambda_\mathbf{0})$ in terms of arc length parameter s by setting

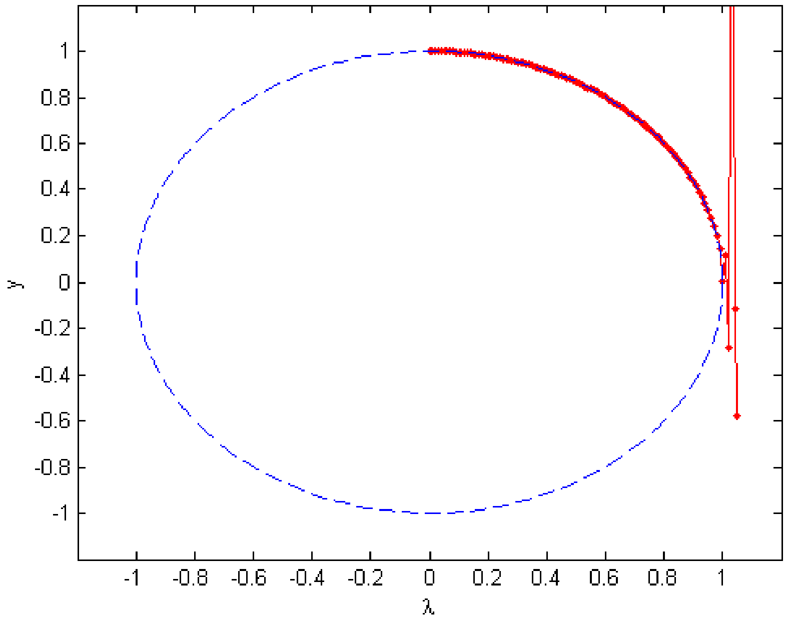


Figure 9: The solution of $F(y, \lambda) = \lambda^2 + y^2 - 1$ using the natural parameter continuation method. The blue dashed line represents the actual solution and the red points are the solution obtained using the method described above. As noted above, the natural parameter continuation method fails to trace the curve at the turning point, $\lambda = 1$ (red dots lie far off the actual solution curve).

$(\mathbf{u_0}, \lambda_\mathbf{0}) = (\mathbf{u}(\mathbf{s_0}), \lambda(\mathbf{s_0}))$. The pseudo arc-length normalization equation is given as

$$N(\mathbf{u}, \lambda, \mathbf{s}) = \mathbf{N_s}(\mathbf{u}, \lambda) = \|(\mathbf{u}, \lambda) - (\mathbf{u}(\mathbf{s_0}), \lambda(\mathbf{s_0}))\|^\mathbf{2} - \mathbf{\Delta s^2} \tag{42}$$

By requiring $N = 0$, we require the "distance" between the new and old solutions to be Δs. We now solve the appended nonlinear system given by,

$$\mathcal{F}(\mathbf{u}, \lambda, \mathbf{s}) = \begin{pmatrix} F(\mathbf{u}, \lambda) \\ N(\mathbf{u}, \lambda, \mathbf{s}) \end{pmatrix} \tag{43}$$

The Jacobian of this system is

$$\begin{bmatrix} \frac{\partial F}{\partial \mathbf{u}} & \frac{\partial F}{\partial \lambda} \\ \frac{\partial N}{\partial \mathbf{u}} & \frac{\partial N}{\partial \lambda} \end{bmatrix} \tag{44}$$

The introduction of arc-length normalization equation allows us to step along a solution curve even when the matrix of derivatives for F, $\frac{\partial F}{\partial \mathbf{u}}$ is singular like at the turning points since the new matrix of derivatives $D\mathcal{F}$ is non-singular at turning points.

We show the implementation of the pseudo arc-length continuation method for our circle example in (41). The pseudo arc-length normalization equation for the circle is

$$N(y, \lambda, s) = (y - y_0)^2 + (\lambda - \lambda_0)^2 - \Delta s^2$$

and the new Jacobian given as

$$D\mathcal{F} = \begin{bmatrix} 2y & 2\lambda \\ 2(y - y_0) & 2(\lambda - \lambda_0) \end{bmatrix}$$

is non singular at all points on curve and hence this algorithm is able to trace the turning points of the curve $F(y, \lambda) = 0$, Figure (10).

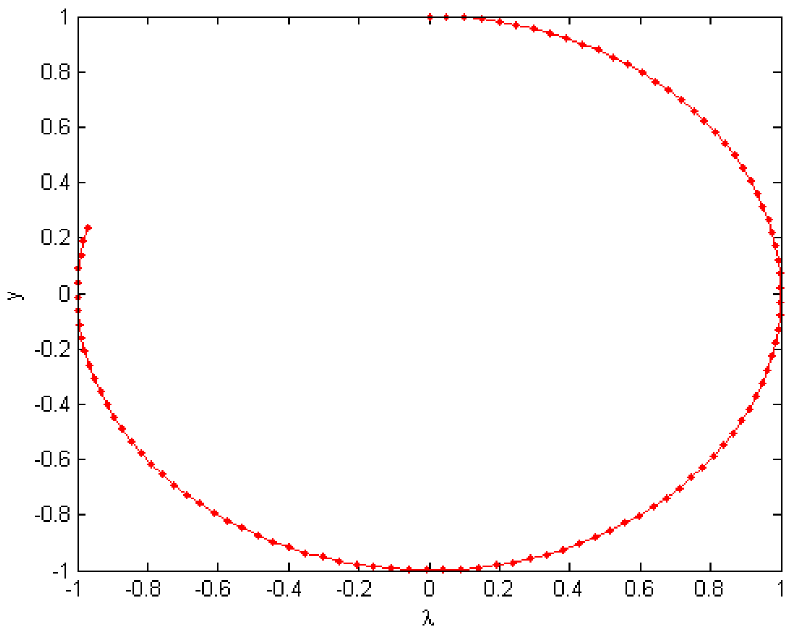


Figure 10: The solution of $F(y, \lambda) = \lambda^2 + y^2 - 1$ using the pseudo arclength parameter continuation method. Red dots represent the solution using pseudo arclength continuation method. This method is able to trace the solution along the turning points.

5 Results

5.1 ODE Model

Recall equation (3),

$$y'' + \delta y' + ay^+ - by^- = \hat{g} + \hat{\lambda}\sin(\mu t).$$

We want to investigate how the periodic solutions to (3) vary with the changes in periodic forcing function. We need a method that looks for periodic solution for different values of parameters. One approach is to treat the boundary value problem (3) as an initial value problem and employ Newton's method solver to find initial conditions which lead to periodic solutions. To employ Newton's method, we search for the ideal initial position and velocity by defining a function of two variables (initial position and velocity) that gives the difference of the starting values and that of the final values when one period has been completed in a ODE solver. The zeros of this new function will be the desired initial condition that yields a periodic solution.

To illustrate this method, let u_0 and v_0 denote the initial position and initial velocity respectively and $G\begin{bmatrix} u \\ v \end{bmatrix}$ denote the position and velocity of a solution of (3) after one time period ($T = 2\pi/\mu$, where μ is the forcing frequency in (3)). Thus, finding periodic solution of (3) is equivalent to finding the zeros of

$$F\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} F_1(u,v) \\ F_2(u,v) \end{bmatrix} = G\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix}. \tag{45}$$

We perform Newtons Method on this system following the iterative scheme,

$$x_{n+1} = x_n - DF^{-1}(x_n)F(x_n) = \begin{bmatrix} u_n \\ v_n \end{bmatrix} - \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix}^{-1} F\begin{bmatrix} u_n \\ v_n \end{bmatrix} \tag{46}$$

We iteratively compute $x_{n+1} = x_n - DF^{-1}(x_n)F(x_n)$ until our error is sufficiently small and our result is the set of initial conditions corresponding to the desired solution. As noted earlier, the natural parameter continuation method fails to trace turning points, see bifurcation diagram in figure (11).

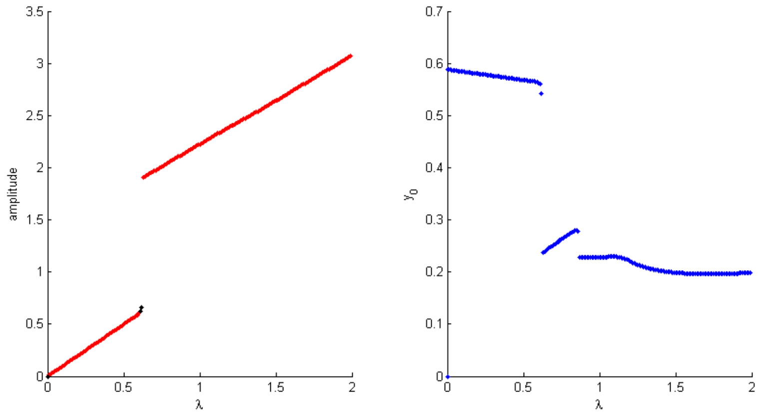


Figure 11: The bifurcation diagram (amplitude versus λ) in the left and plot of initial position generating periodic solution versus λ in the right. In the bifurcation diagram (left), red points corresponds to the solutions for which Newtons method converged and black dots refer to the solutions where Newtons method didn't converge.

To complete the bifurcation curve, we use the pseudo arc-length continuation algorithm for (3). We proceed as follow:

1. If (u^0, v^0, λ^0) is a solution of F, we parametrize (u^0, v^0, λ^0) in terms of arc length parameter s by setting $(u^0, v^0, \lambda^0) = (u(s_0), v(s_0), \lambda(s_0))$.

2. Introduce the pseudo arclength normalization equation (47) dependent on initial conditions (u, v), λ and the arc length parameter (s).

$$N(u, v, \lambda, s) = N_s(u, v, \lambda) = \|(u, v, \lambda) - (u(s_0), v(s_0), \lambda(s_0))\|^2 - \Delta s^2 \tag{47}$$

3. Evaluate the Jacobian matrix of the appended system

$$\mathcal{F}(u, v, \lambda, s) = \begin{pmatrix} F(u, v, \lambda) \\ N(u, v, \lambda, s) \end{pmatrix}. \tag{48}$$

using

$$D\mathcal{F} = \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial \lambda} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial \lambda} \\ \frac{\partial N}{\partial u} & \frac{\partial N}{\partial v} & \frac{\partial N}{\partial \lambda} \end{bmatrix} \tag{49}$$

4. Compute amplitude for the equilibrium (initial) solution $(u_0, v_0, \lambda_0) = (u(s_0), v(s_0), \lambda(s_0))$ by running the ODE solver for one time period.

5. Run the first iteration using natural parameter continuation algorithm (45-46) with $s = 0$ and $\lambda = \lambda + \Delta s$ to obtain (u_1, v_1, λ_1) using (u_0, v_0, λ_0) as initial guess.

6. After second iteration, increment arclength parameter s by Δs

$$s_i = s_{i-1} + \Delta s.$$

7. Solve for the roots of (48) using Newtons method following the iterative scheme

$$x_{n+1} = x_n - D\mathcal{F}^{-1}(x_n)\mathcal{F}(x_n) \tag{50}$$

where $x_i = (u_i, v_i, \lambda_i)$ until the error is sufficiently small using secant predictor. The secant predictor for solution set $\mathbf{u}$ at $t = i$ is given by

$$x_{i-1} + (x_{i-1} - x_{i-2}).$$

For our model this is

$$2(u_{i-1}, v_{i-1}, \lambda_{i-1}) - (u_{i-2}, v_{i-2}, \lambda_{i-2}).$$

8. Calculate amplitude for each periodic solution.

9. Increment s using $s_{i+1} = s_i + \Delta s$ and repeat steps 7 and 8.

The pseudo arc-length algorithm is able to trace turning points on the bifurcation curve, see figure (12) unlike the natural parameter continuation method. But $\mathcal{F}$ may still

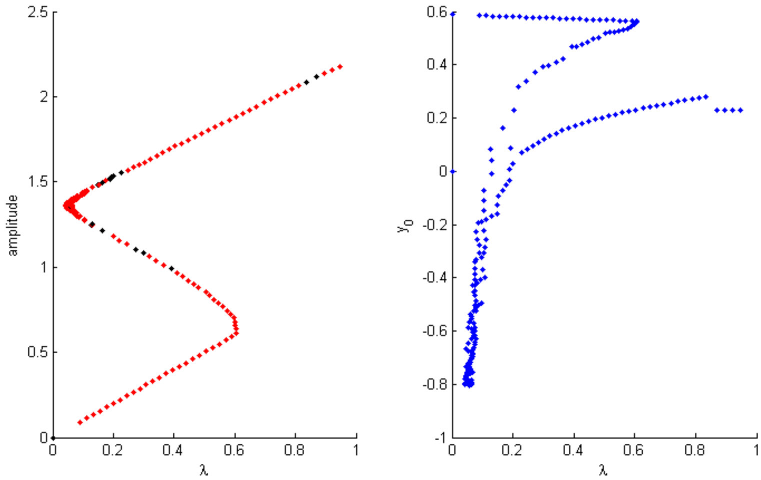


Figure 12

fail at points where a bifurcation into a new branches of solutions occurs and it cannot detect where such new branches occur. It can however, generate bifurcation curves in both forward and backward direction. Figures (13)-(15) show some solutions of (3) pertaining to different amplitudes. We notice that (3) has multiple solutions for small values of λ.

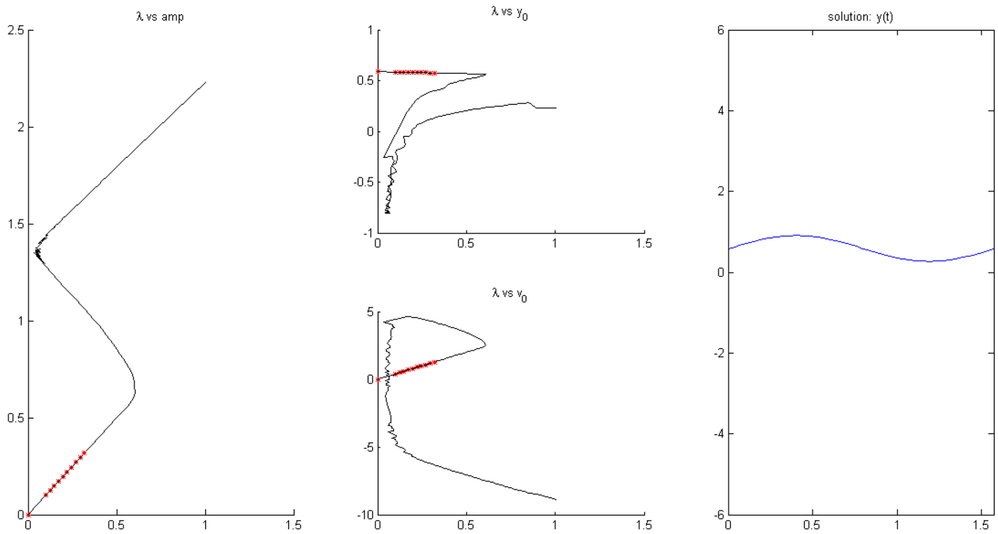


Figure 13: The figure shows the bifurcation diagram (right), initial position,y_0 (center top), initial velocity,v_0 (center bottom) and the periodic solution (left) corresponding to different λ values.

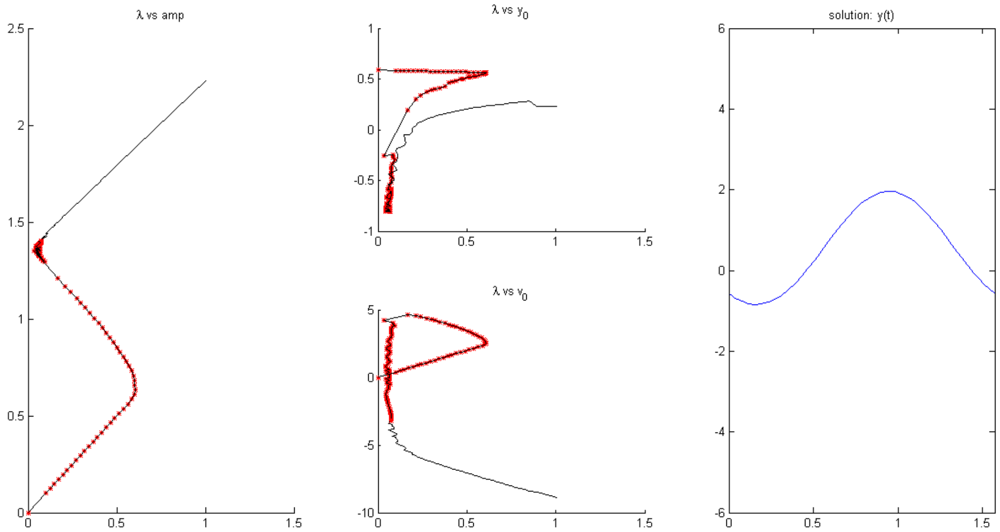


Figure 14: The figure shows the bifurcation diagram (right), initial position,y_0 (center top), initial velocity,v_0 (center bottom) and the periodic solution (left) corresponding to different λ values.

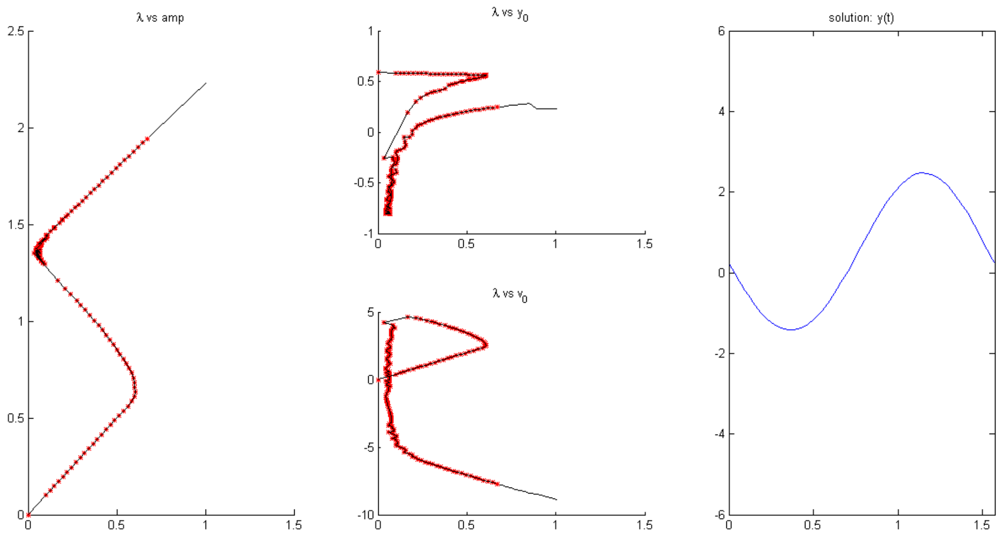


Figure 15: The figure shows the bifurcation diagram (right), initial position,y_0 (center top), initial velocity,v_0 (center bottom) and the periodic solution (left) corresponding to different λ values.

5.2 PDE Model

We will start with the finite difference scheme for the fourth derivative u_{xxxx}. We use the five point centered difference method to approximate the fourth derivative of u. Starting with the Taylor expansions,

$$\begin{aligned}
u(x) &= u(\tilde{x}) + u^{(1)}(\tilde{x})(x-\tilde{x}) + \frac{(x-\tilde{x})^2}{2!}u^{(2)}(\tilde{x}) + \frac{(x-\tilde{x})^3}{3!}u^{(3)}(\tilde{x}) + \frac{(x-\tilde{x})^4}{4!}u^{(4)}(\tilde{x}) + O(x-\tilde{x})^5 \\
u(\tilde{x}-2h) &= u(\tilde{x}) - 2hu^{(1)}(\tilde{x}) + \frac{4h^2}{2!}u^{(2)}(\tilde{x}) - \frac{8h^3}{3!}u^{(3)}(\tilde{x}) + \frac{16h^4}{4!}u^{(4)}(\tilde{x}) + O(h^5) \\
u(\tilde{x}-h) &= u(\tilde{x}) - hu^{(1)}(\tilde{x}) + \frac{h^2}{2!}u^{(2)}(\tilde{x}) - \frac{h^3}{3!}u^{(3)}(\tilde{x}) + \frac{h^4}{4!}u^{(4)}(\tilde{x}) + O(h^5) \\
u(\tilde{x}+h) &= u(\tilde{x}) + hu^{(1)}(\tilde{x}) + \frac{h^2}{2!}u^{(2)}(\tilde{x}) + \frac{h^3}{3!}u^{(3)}(\tilde{x}) + \frac{h^4}{4!}u^{(4)}(\tilde{x}) + O(h^5) \\
u(\tilde{x}+2h) &= u(\tilde{x}) + 2hu^{(1)}(\tilde{x}) + \frac{4h^2}{2!}u^{(2)}(\tilde{x}) + \frac{8h^3}{3!}u^{(3)}(\tilde{x}) + \frac{16h^4}{4!}u^{(4)}(\tilde{x}) + O(h^5)
\end{aligned} \tag{51}$$

Collecting the terms, we get

$$\begin{aligned}
D^4u(\tilde{x}) &= (a+b+c+d+e)u(\tilde{x}) + (-2b-c+d+2e)hu'(\tilde{x}) + \frac{h^2}{2}(4b+c+d+4e)u''(\tilde{x}) \\
&\quad + \frac{h^3}{6}(-8b-c+d+8e)u^3(\tilde{x}) + \frac{h^4}{24}(16b+c+d+16e)u^4(\tilde{x}) + \cdots
\end{aligned} \tag{52}$$

For this to agree with $u^{(4)}(\tilde{x})$, up to order $O(h^2)$, we need

$$\begin{aligned}
a+b+c+d+e &= 0 \\
16b+c+d+16e &= \frac{24}{h^4} \\
-8b-c+d+8e &= 0 \\
4b+c+d+4e &= 0 \\
-2b-c+d+2e &= 0
\end{aligned} \tag{53}$$

Solving the linear system (53) gives

$$a = \frac{6}{h^4}, \quad b = e = \frac{1}{h^4}, \quad c = d = -\frac{4}{h^4}$$

Thus, our second order five point centered difference approximation for fourth derivative is

$$D^4u(\tilde{x}) = \frac{1}{h^4}\big[u(\tilde{x}-2h) - 4u(\tilde{x}-h) + 6u(\tilde{x}) - 4u(\tilde{x}+h) + u(\tilde{x}+2h)\big] \tag{54}$$

We now select N interior grid points and approximate the solution at discrete points x_i for $i = 0, 1, \cdots, N+1$, where x_0 and x_{N+1} are the boundary points. Using (54) to discretize u_{xxxx} at each x_i, gives

$$u_{xxxx}(x_i) = \frac{1}{h^4}\Big(u(x_{i-2}) - 4u(x_{i-1}) + 6u(x_i) - 4u(x_{i+1}) + u(x_{i+2})\Big) + O(h^2), \quad i = 1, 2\cdots 1, 2, \cdots, N$$

Writing the individual equation for each x_i, gives

$$\begin{aligned}
\frac{1}{h^4}\Big(u(x_{-1}) - 4u(x_0) + 6u(x_1) - 4u(x_2) + u(x_3)\Big) &= f(x_1)\\
\frac{1}{h^4}\Big(u(x_0) - 4u(x_1) + 6u(x_2) - 4u(x_3) + u(x_4)\Big) &= f(x_2)\\
\frac{1}{h^4}\Big(u(x_1) - 4u(x_2) + 6u(x_3) - 4u(x_4) + u(x_5)\Big) &= f(x_3)\\
&\vdots\\
\frac{1}{h^4}\Big(u(x_{N-3}) - 4u(x_{N-2}) + 6u(x_{N-1}) - 4u(x_N) + u(x_{N+1})\Big) &= f(x_{N-1})\\
\frac{1}{h^4}\Big(u(x_{N-2}) - 4u(x_{N-1}) + 6u(x_N) - 4u(x_{N+1}) + u(x_{N+2})\Big) &= f(x_N)
\end{aligned} \tag{55}$$

We use the boundary condition to eliminate the "ghost points" such that $u(x_{-1})$ and $u(x_{N+2})$ from the equation. For the purpose of illustration, let's consider boundary condition where both ends are hinged,

$$\begin{aligned}
u(x_0) &= 0\\
u_{xx}(x_0) &= \frac{u(x_{-1}) - 2u(x_0) + u(x_1)}{h^2} = 0\\
\Rightarrow u(x_{-1}) &= -u(x_1)
\end{aligned}$$

$$
\begin{aligned}
&u(x_{N+1}) = 0\\
&u_{xx}(x_{N+1}) = \frac{u(x_N) - 2u(x_{N+1}) + u(x_{N+2})}{h^2} = 0\\
&\Rightarrow u(x_{N+2}) = -u(x_N)
\end{aligned}
$$

We can represent (55) as

$$MU = f(x,t) \quad \text{where}$$

$$
M = \begin{bmatrix}
5 & -4 & 1 & & & & & \\
-4 & 6 & -4 & 1 & & & & \\
1 & -4 & 6 & -4 & 1 & & & \\
 & 1 & -4 & 6 & -4 & 1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & 1 & -4 & 6 & -4 & 1 \\
 & & & & 1 & -4 & 6 & -4 \\
 & & & & & 1 & -4 & 5
\end{bmatrix} \tag{56}
$$

and $U = [u_1, u_2, \cdots, u_N]^T$

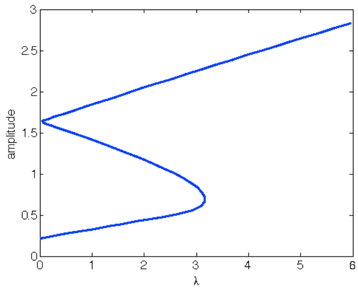
We begin by breaking down (2) into system of two equations as follow:

$$
\begin{aligned}
U_t &= V\\
V_t &= U_{tt} = -MU - dIV + F(U,V,t)
\end{aligned} \tag{57}
$$

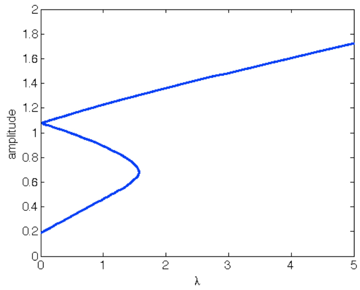
where M is the matrix from (56), I is an identity matrix and $F(U,V,t) = -bU^+ + g + f(x,t)$ represents the non-linear part of the equation. We can further write (57) as

$$\mathcal{U}_t = B\mathcal{U} + G(\mathcal{U}, t) \tag{58}$$

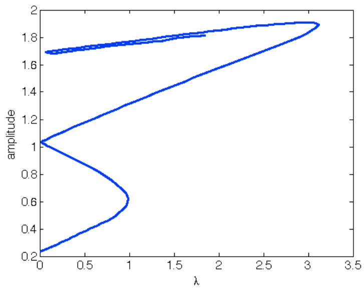
where $\mathcal{U} = \begin{bmatrix} U \\ V \end{bmatrix}$, $B = \begin{bmatrix} O & I \\ -M & -dI \end{bmatrix}$ and $G(\mathcal{U},t) = \begin{bmatrix} O \\ F(U,V,t) \end{bmatrix}$.



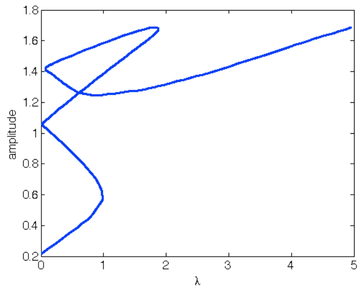
(a) Both ends clamped



(b) Both ends hinged



(c) One end clamped one free



(d) One end hinged one free

We solve (58) as an initial value problem using IMEX method (59) to march in time

$$\frac{\mathcal{U}^{k+1}-\mathcal{U}^{k}}{\Delta t} = \frac{1}{2}\Big(B\mathcal{U}^{k+1}+B\mathcal{U}^{k}\Big)+\frac{3}{2}G(t^{k},\mathcal{U}^{k})-\frac{1}{2}G(t^{k-1},\mathcal{U}^{k-1})$$

$$\boxed{\mathcal{U}^{k+1} = \Big(I-\frac{\Delta t}{2}B\Big)^{-1}\left[\Big(I+\frac{\Delta t}{2}B\Big)\mathcal{U}^{k}+\Delta t\left(\frac{3}{2}G(t^{k},\mathcal{U}^{k})-\frac{1}{2}G(t^{k-1},\mathcal{U}^{k-1})\right)\right]} \qquad (59)$$

for one time period and employ Newtons method on the system appended with arc-length normalization equation to search for initial values that gives periodic solution. This method is called an IMEX method for IMplicit/EXplicit. It is a combination of the implicit Crank Nicolson method in the linear part, and the explicit second-order Adams-Bashforth method in the nonlinear part. The procedure is similar to the one described for the ODE system which is omitted for the sake of simplicity. We represent the dependence of solutions on the parameter λ in the above bifurcation diagrams obtained by plotting the amplitude of the solution versus the forcing amplitude λ. The bifurcation diagram for various boundary conditions are shown in the above figure. These diagrams provide numerical evidence of multiple (three in (a), (b) and five in (c), (d)) solutions to (2) for a range of λ values near $\lambda = 0$.

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References

[1] Chenkui Zhong Yukun An. Periodic solutions of a nonlinear suspension bridge equation with damping and nonconstant load. *Journal of Mathematical Analysis and Applications*, 279, 2003.

[2] P.J.McKenna A.C.Lazer. Non-linear flexing in a periodically forced floating beam. *Mathematical Methods in the Applied Sciences*, 14, 1991.

[3] K. M. O'Neill L. D. Humphreys, P. J. McKenna. High frequency shaking induced by low frequency forcing: periodic oscillations in a spring-cable system. *Nonlinear Analysis. Real World Applications*, 11(5):4312–4325, 2010.