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Terje Hill
Florida Atlantic University

David Robbins
Trinity College, david.robbins@trincoll.edu

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Character amenability of vector-valued algebras

Terje Hill and David A. Robbins

Abstract. Let $\{A_x : x \in X\}$ be a collection of complex Banach algebras indexed by the compact Hausdorff space X. We investigate the character amenability of certain algebras $\mathcal{A}$ of A_x-valued functions in relation to the character amenability of the A_x.

1. Preliminaries

Suppose that X is a set (usually, a topological space), and that $\{A_x : x \in X\}$ is a collection of Banach spaces (algebras) indexed by X, over a common scalar field $\mathbb{K}$, either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A} \subset \prod\{A_x : x \in X\}$ be a Banach space (algebra) of functions σ, so that $\sigma(x) \in A_x$ for all $x \in X$. If $\mathcal{A}$ has a property $\mathcal{P}$ whenever each A_x has property $\mathcal{P}$ (or maybe when only some do), we say that $\mathcal{P}$ is hereditary for $\mathcal{A}$. It seems reasonable to ask for which spaces $\mathcal{A}$ and properties $\mathcal{P}$ this hereditary condition holds.

This subject has been addressed in a number of papers over the years, with conditions as follows. We take X to be a compact Hausdorff space and, as above, we let $\mathcal{A} \subset \prod\{A_x : x \in X\}$ be a vector space which satisfies the following:

C1) for each $x \in X, A_x = p_x(\mathcal{A}) = \{\sigma(x) : \sigma \in \mathcal{A}\}$ ($\mathcal{A}$ is said to be full; p_x is the evaluation map at x);

C2) for each $\sigma \in \mathcal{A}$, the norm map $x \mapsto \|\sigma(x)\|$ is upper semicontinuous (and hence σ is bounded) on X;

C3) $\mathcal{A}$ is a $C(X)$-module under the pointwise operations;

C4) $\mathcal{A}$ is complete in the sup-norm (for $\sigma \in \mathcal{A}, \|\sigma\| = \sup\{\|\sigma(x)\| : x \in X\}$); and

C5) if each A_x is a Banach algebra, then $\mathcal{A}$ is also closed under pointwise multiplication (and hence is a Banach algebra).

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We call such a space $\mathcal{A}$ which satisfies C1) - C4) an upper semicontinuous function space (algebra, if C5) also holds) with fibers A_x, and abbreviate to just "function space" or "function algebra." Then $\mathcal{B}$, $\mathcal{C}$, etc. will denote such function spaces, with fibers B_x, C_x, etc. This is based on the language of [1], which called these spaces "upper semicontinuous function modules." The study of such upper semicontinuous function spaces dates back many years; see the end of Section 2 of [6] for a brief (and quite incomplete) bibliographical note.

Examples of such function spaces (algebras) $\mathcal{A}$ can be found (using slightly different language) in [6, Section 2] and its references, and also in [4] and [12]. Particular examples of such spaces $\mathcal{A}$ are the following.

Example 1. If X is compact, a $C(X)$-module M is said to be $C(X)$-locally convex if (among other equivalent formulations) whenever $f, g \in C(X)$ with $fg = 0$ and $m \in M$, then $\|(f+g)m\| = \max\{\|fm\|, \|gm\|\}$; see [4, Theorem 7.14]. It is easy to check that a space $\mathcal{A}$ satisfying the conditions C1) - C4) above is $C(X)$-locally convex. Conversely, suppose M is $C(X)$-locally convex. For $x \in X$, let I_x be the maximal ideal in $C(X)$ consisting of functions which vanish at x, and let $A_x = M/I_xM$, where I_xM is the closed span in M of elements of the form fm ($f \in I_x, m \in M$). Then M is $C(X)$-isometrically isomorphic to a space of type $\mathcal{A}$, where $m \in M$ maps to $\widehat{m} \in \mathcal{A}$, $\widehat{m}(x) = m + I_xM \in A_x$ (see [4, Theorem 5.9]).

Example 2. Let X be compact and A a Banach space (algebra). Then $\mathcal{A} = C(X, A)$, the space of continuous A-valued functions on X, is a function space (algebra) with $A_x = A$ for all x. If X is locally compact, and we let $X_\infty = X \cup \{\infty\}$ be its standard one-point compactification, then we can identify $C_0(X, A)$, the space of continuous A-valued functions vanishing at infinity, with the function space (and $C(X_\infty)$-module) $\mathcal{B}$ consisting of the continuous A-valued functions on X_∞ which vanish at ∞; the fibers B_x of $\mathcal{B}$ are A_x, if $x \neq \infty$, and $B_\infty = \{0\}$.

We note some important properties of function spaces $\mathcal{A}$.

I) Let $x \in X$, and let B_x be a closed subspace (ideal) of A_x. Then $\mathcal{B}_x = \{\sigma \in \mathcal{A} : \sigma(x) \in B_x\}$ is a closed subspace (ideal) and $C(X)$-submodule of $\mathcal{A}$. The fibers of $\mathcal{B}_x$ are A_y, if $y \neq x$, and B_x. (Proof of the last assertion: Suppose that $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively, let $a_y \in A_y, b_x \in B_x$, and let $\sigma, \tau \in \mathcal{A}$, with $\sigma(y) = a_y$ and $\tau(x) = b_x$. Let $f, g \in C(X)$ be supported on U and V, respectively, with $f(y) = 1$ and $g(x) = 1$. Because $\mathcal{B}_x$ is a $C(X)$-submodule of $\mathcal{A}$, then $f\sigma + g\tau \in \mathcal{B}_x$, and we have $(f\sigma + g\tau)(y) = a_y$ and $(f\sigma + g\tau)(x) = b_x$. Thus, $p_x(\mathcal{B}_x) = B_x$ and $p_y(\mathcal{B}_x) = A_y$.)

II) If $\mathcal{A}$ is a function algebra, and if each A_x contains a bounded approximate identity $\{u_{\lambda_x}\}$ ($\lambda_x \in \Lambda_x$), and these approximate identities are uniformly bounded (i.e. for some m, all $\lambda_x \in \Lambda_x$, and all Λ_x, we have

$\|u_{\lambda_x}\| \leq m$), then $\mathcal{A}$ also has an approximate identity of the same bound; this is the content of Corollary 2.3 and its preceding Lemma in [9]. We will prove later that if each A_x contains merely an approximate identity, then $\mathcal{A}$ also has an approximate identity.

Previous work has shown, for example, that under reasonable boundedness conditions, both amenability (see [9]) and module amenability (see [6]) are hereditary for function algebras $\mathcal{A}$; this motivates a search for an analogous boundedness condition for character amenability. Going back farther, and taking $\mathcal{A}$ to be simply a function space with fibers A_x, then $\mathcal{A}$ satisfies the approximation property if and only if each A_x does so also ([3]). Other examples, e.g. some hereditary geometric properties, can be found in [15], [7], and [8]. Below, in Section 2, we will recall some definitions and properties relating to character amenability for a Banach algebra A, describe what we mean for these properties to be hereditary for vector-valued function algebras of type $\mathcal{A}$, and obtain conditions sufficient for the properties to be hereditary for $\mathcal{A}$.

2. Character amenability and heritability

We first establish some vocabulary and notation. Recall the notion of amenability for Banach algebras, as introduced in [10]. If A is a complex Banach algebra, and if E is a Banach A-bimodule, then a bounded linear map $D : A \to E$ is called a *derivation* if $D(ab) = aD(b) + D(a)b \in E$ for all $a, b \in A$. If a derivation $D : A \to E$ has the form

$$D(a) = \delta_m(a) = am - ma$$

for some $m \in E$, then D is said to be *inner*. If E is such a bimodule, then E^* can be made into a Banach A-bimodule in a standard fashion via the actions

$$\langle am^*, m\rangle = \langle m^*, ma\rangle \text{ and } \langle m^*a, m\rangle = \langle m^*, am\rangle$$

for all $a \in A, m \in E$, and $m^* \in E^*$. We then say that A is *amenable* if for any A-bimodule E, and every derivation $D : A \to E^*$, we have $D = \delta_{m^*}$ for some $m^* \in E^*$. We note that amenability of A can also be expressed in homological terms, the details of which are not necessary here.

Over the years, there have been many variations and generalizations of the idea of amenability. Examples are weak amenability, module amenability and, the one on which we will focus here, character amenability, as defined in [11].

Let A be a complex Banach algebra, and let $\phi \neq 0$ be a character of A, i.e. an algebra homomorphism $\phi : A \to \mathbb{C}$. Then ϕ is automatically continuous, and $\ker\phi$ is a (maximal) closed ideal in A of codimension 1. If E is a right

Banach A-module, and ϕ is a character on A, define a left action of A on E by

$$a * m = \phi(a)m$$

for $a \in A$ and $m \in E$, thus making E a Banach A-bimodule, denoted by E_ϕ; as sets, we have $E = E_\phi$. The right action $*$ of $a \in A$ on $m^* \in E_\phi^*$ is thus given by

$$\langle m^* * a, m \rangle = \langle m^*, a * m \rangle = \langle m^*, \phi(a)m \rangle = \langle \phi(a)m^*, m \rangle .$$

Following [11], we say that A is *ϕ-amenable* if for any right Banach A-module E and any continuous derivation $D : A \to E_\phi^*$ there exists $m^* \in E_\phi^*$ such that for all $a \in A$ we have

$$D(a) = \delta_{m^*}(a) = am^* - m^* * a = am^* - \phi(a)m^*.$$

Then, denoting the space of all characters on A by $\Delta(A)$, A is said to be *character amenable* if A is ϕ-amenable for each character $\phi \in \Delta(A)$.

As in [14], for $\phi \in \Delta(A)$ we say that A is *locally approximately ϕ-amenable* if for any right A-module E and any continuous derivation $D : A \to E_\phi^*$ there is a net $\{\xi_\lambda : \lambda \in \Lambda\} \subset E^*$ such that for each $a \in A$ the inner derivations $\delta_{\xi_\lambda}(a) = a\xi_\lambda - \xi_\lambda * a = a\xi_\lambda - \phi(a)\xi_\lambda$ converge in norm to $D(a)$ in E_ϕ^*.

We call a net $\{u_\gamma : \gamma \in \Gamma\} \subset A$ a *locally approximate ϕ-mean* if $\phi(u_\gamma) = 1$ for all γ, and $\|u_\gamma a - \phi(a)u_\gamma\| \to 0$ for all $a \in A$. If the locally approximate ϕ-mean is uniformly bounded, i.e. if there exists m such that $\|u_\gamma\| \leq m$ for all γ, then we call $\{u_\lambda\}$ a *bounded approximate ϕ-mean.*

Especially, note that if $\{u_\lambda\}$ is a locally approximate ϕ-mean for the algebra A, then for $u \in \{u_\lambda\}$ we have $u + \ker\phi$ is the identity in $A/\ker\phi$, i.e. $ua + \ker\phi = a + \ker\phi = au + \ker\phi$ for all $a \in A$. In particular, we see that $\|u_\lambda + \ker\phi\|$ is constant across all $\lambda \in \Lambda$.

It is the case that, in a fashion similar to amenability in its classic definition (see e.g. [5, p. 254]), there are intrinsic characterizations of local approximate ϕ-amenability and ϕ-amenability. We have the following, taken from [14] and its references.

Proposition 1 ([14], Lemmas 1.1 and 1.2)**.** *Let A be a complex Banach algebra, and suppose $\phi \in \Delta(A)$. The following are equivalent:*
1) *A is ϕ-amenable;*
2) *A has a bounded approximate ϕ-mean; and*
3) $\ker\phi$ *has a bounded approximate right identity.*

Proposition 2 ([14], Lemmas 1.1 and 1.2)**.** *Let A be a complex Banach algebra, and suppose that $\phi \in \Delta(A)$. The following are equivalent:*
1) *A is locally approximately ϕ-amenable;*
2) *A has a (possibly unbounded) locally approximate ϕ-mean; and*
3) $\ker\phi$ *has a (possibly unbounded) right approximate identity.*

In the following, $\mathcal{A}$ will always be a function algebra (i.e. a function space which is also a Banach algebra) over the compact Hausdorff space X, with fibers A_x (which are themselves Banach algebras).

We need to specify the notion of hereditability of (locally approximate) character amenability that we will explore, and use the following short development to obtain the "correct" such notion.

Let B be a complex Banach algebra. Recall that a representation of B is a continuous algebra homomorphism $T : B \to L(Z)$, where Z is some Banach space and $L(Z)$ denotes the space of continuous operators from Z to itself. The representation T is said to be irreducible provided that $T \neq 0$ and that, for $b \in B$, the only invariant subspaces in Z for $T(b)$ are $\{0\}$ and Z itself.

Translating the language of Proposition 1 of [13] to our current situation, we can then identify the characters of a function algebra. Again, $p_x : \mathcal{A} \to A_x, \sigma \mapsto \sigma(x)$, is the evaluation map at $x \in X$.

Lemma 1 ([13], Prop. 1)**.** *Let $\mathcal{A}$ be a function algebra, and suppose that for some Banach space Z, $T : \mathcal{A} \to L(Z)$ is an irreducible representation of $\mathcal{A}$. Then there exists unique $x \in X$ and an irreducible representation $S_x : A_x \to L(Z)$ such that $T = S_x \circ p_x$.*

Lemma 2. *Let B be a complex Banach algebra, and let $\phi \in \Delta(B)$. Then ϕ is an irreducible representation of B onto $L(\mathbb{C}) = \mathbb{C}$.*

Proof. For $b \in B$, we can interpret $\phi(b)$ as an element of $L(\mathbb{C}) = \mathbb{C}$: if $\alpha \in \mathbb{C}$, the action of $\phi(b)$ on α is defined by $[\phi(b)](\alpha) = \phi(b) \cdot \alpha$, where $\cdot$ is scalar multiplication. Evidently, the only invariant subspaces of $\mathbb{C}$ under $\phi(b)$ are $\{0\}$ and $\mathbb{C}$ itself. $\square$

We then apply these lemmas to obtain a complete description of the elements of $\Delta(\mathcal{A})$.

Corollary 1. *Let $\mathcal{A}$ be a function algebra, and suppose that $H \in \Delta(\mathcal{A})$. Then there exists a unique $x \in X$ and $\phi_x \in \Delta(A_x)$ such that $H = H_{\phi_x} = \phi_x \circ p_x$. Conversely, if $\phi_x \in \Delta(A_x)$, then $H_{\phi_x} = \phi_x \circ p_x \in \Delta(\mathcal{A})$.*

As a result of the immediately preceding Corollary, we can identify $\Delta(\mathcal{A})$ with $\dot{\bigcup}\{\Delta(A_x) : x \in X\}$, the disjoint union of the character spaces of the A_x. We note that this is one of a number of similar such results for spaces of vector-valued functions involving the upper semicontinuity of the norm (for Banach spaces) or seminorms (for where the function spaces take values in locally convex topological vector spaces).

It is then easy to identify the kernel of $H_{\phi_x} = \phi_x \circ p_x \in \Delta(\mathcal{A})$: we have $H_{\phi_x}(\sigma) = 0$ if and only if $\phi_x(\sigma(x)) = 0$, i.e. $\sigma \in \ker H_{\phi_x} = \ker(\phi_x \circ p_x) \subset \mathcal{A}$ if and only if $\sigma(x) \in \ker \phi_x = K_{\phi_x} \subset A_x$. Using the notation from Section 1, set $\mathcal{K}_{\phi_x} = \{\sigma \in \mathcal{A} : \sigma(x) \in K_{\phi_x}\} = \ker H_{\phi_x}$. Note that $\|H_{\phi_x}\| = \|\phi_x\|$.

Corollary 1 then leads us to the notion of hereditability for (locally approximate) character amenability we will explore: If $\mathcal{A}$ is a function algebra, and if $\phi_x \in \Delta(A_x)$ is such that A_x is (locally approximately) ϕ_x-amenable, we will look for conditions sufficient to guarantee that $\mathcal{A}$ is (locally approximately) $H_{\phi_x} = (\phi_x \circ p_x)$-amenable.

Now, Propositions 1 and 2 tell us that we might want to look for (bounded) right approximate identities in $\mathcal{K}_{\phi_x} = \ker H_{\phi_x} = \{\sigma \in \mathcal{A} : \sigma(x) \in \ker \phi_x\}$. But $\mathcal{K}_{\phi_x}$ is itself a function algebra with fibers A_y ($y \neq x$) and K_{ϕ_x}. Since, for example, a function algebra will have a bounded right approximate identity if and only if its fibers have uniformly bounded right approximate identities (see II) of Section 1), it is then reasonable to look at the kinds of fiberwise right approximate identities that might exist.

As a model for the criteria we are seeking, we recall how amenability in its classic sense can be inherited for a function algebra $\mathcal{A}$. Specifically, a complex algebra A is amenable if and only if it has a bounded approximate identity and the kernel of the multiplication map $\mu : A\widehat{\otimes}A^{op} \to A$, $a \otimes b \mapsto ab$ has a bounded right approximate identity, where A^{op} is A with the multiplication reversed. Then a function algebra $\mathcal{A}$ is amenable if and only the fibers A_x have uniformly bounded approximate identities and the kernels of the maps $\mu_x : A_x\widehat{\otimes}A_x^{op} \to A_x$ have uniformly bounded right approximate identities; the A_x are then said to be uniformly amenable. (See [9] for this condition, and [5] for the equivalence of amenability and the existence of the two noted approximate identities.) Given the crucial role that uniform boundedness plays in the inheritability of classical amenability, we are led in this instance to seek an analogue for character amenability; in particular, to find a condition involving character amenability of fibers which involves some idea of uniform boundedness.

As we proceed, in order to avoid an uninstructive complication, we will assume that $\mathcal{A}$ is a function algebra over X such that if $\Delta(A_x) = \varnothing$, then $A_x = \{0\}$. From II) of Section 1 (and, looking ahead to Lemma 5), we can see that the existence of (bounded) approximate identities for any function algebra $\mathcal{A}$ depends only on the existence of (uniformly bounded) approximate identities for $\{A_x : A_x \neq \{0\}\}$.

Definition 1. Let $\mathcal{A}$ be a function algebra over X, and let $Y = \{y \in X : A_y \neq \{0\}\}$. Suppose that there exists a collection $\{\phi_y \in \Delta(A_y) : y \in Y\}$ such that each A_y is ϕ_y-character amenable, with associated bounded right approximate identity $\{a_{\gamma_y} : \gamma_y \in \Gamma_y\} \subset \ker \phi_y$ and associated bounded approximate ϕ_y-mean $\{u_{\lambda y} : \lambda_y \in \Lambda_y\}$, and that

1) the right approximate identities are uniformly bounded by m; and

2) the approximate ϕ_y-means are uniformly bounded by q.

Then we say that $\Phi = \{\phi_y : y \in Y\}$ is a collection of *uniformly amenable characters* for the A_y.

The following result, and some variants, are crucial to our determination of the existence of (bounded) approximate identities in kernels of elements of $\Delta(\mathcal{A})$.

Proposition 3 ([2], Prop. 7.1)**.** *Let B be a Banach algebra, and let $I \subset B$ be a closed two-sided ideal. If I has a bounded (by q) (right) approximate identity, and if B/I has a bounded (by m) (right) approximate identity, then B has an approximate identity bounded by $m + q + mq$.*

Lemma 3. *Suppose that A is a complex algebra, and that $\phi \in \Delta(A)$ is such that A is ϕ-amenable. Then A has a bounded right approximate identity.*

Proof. From Proposition 1 we know that $\ker \phi$ has an associated bounded right approximate identity $\{a_\gamma\}$ and that A has a bounded approximate ϕ-mean $\{u_\lambda\}$. Let m be a bound for $\{a_\gamma\}$ and q a bound for $\{u_\lambda\}$. Then, for $u \in \{u_\lambda\}$, note that $u + \ker \phi$ is the identity in $A/\ker \phi$, and that $\|u + \ker \phi\| \leq \|u\| \leq q$. It then follows from the preceding proposition that A has a bounded right approximate identity of norm at most $m + q + mq$. □

Theorem 1. *Let $\mathcal{A}$ be a function algebra, and fix $x \in X$. Let $\psi_x \in \Delta(A_x)$, and suppose that A_x is ψ_x-amenable. Suppose there exists a collection Φ of uniformly amenable characters for $\{A_y : A_y \neq \{0\}\}$. Then $\mathcal{A}$ is $H_{\psi_x} = (\psi_x \circ p_x)$-amenable. Therefore, if there exists such a collection Φ, and if each A_x is character amenable, then so is $\mathcal{A}$.*

Proof. We need to show that $\mathcal{K}_{\psi_x} = \ker H_{\psi_x}$ has a bounded right approximate identity. Recall that $\mathcal{K}_{\psi_x}$ is a function algebra with fibers K_{ψ_x} and A_y ($y \neq x$), so that it will suffice to show that the fibers have uniformly bounded right approximate identities.

Consider $A_y \neq \{0\}$, where $y \neq x$. Since Φ is a uniformly amenable collection of characters, there is a uniform bound for the right approximate identities for $\ker \phi_y$, where $\phi_y \in \Phi$, $y \neq x$. Moreover, K_{ψ_x} has a bounded right approximate identity; let m be a uniform bound for all these right approximate identities (in K_{ϕ_y} and K_{ψ_x}). Now, again since Φ is a uniformly amenable set of characters, the respective identities in the quotients $A_y/\ker \phi_y$ are uniformly bounded. (Because, if $\{u_{\lambda_y}\}$ ($y \neq x$) are the uniformly bounded (by q, say) locally approximate ϕ_y-means ($\phi_y \in \Phi$), then, for any $u_y \in \{u_{\lambda_y}\}$, we have

$$\|u_y + \ker \phi_y\| \leq \|u_y\| \leq q.)$$

It follows from Lemma 3 that all fibers A_y ($y \neq x$) have bounded right approximate identities of norm at most $m + q + mq$. Since K_{ψ_x} also has a bounded right approximate identity, it follows from [9] that $\mathcal{K}_{\psi_x}$ has a bounded right approximate identity, and hence that $\mathcal{A}$ is H_{ψ_x}-amenable.

The final assertion follows by what it means for an algebra to be character amenable; see the discussion preceding Proposition 1 in which ϕ-amenability is described. □

Corollary 2. *Let A be a Banach algebra with $\Delta(A) \neq \varnothing$, and let $\mathcal{A}$ be a function algebra such that, for all $x \in X$, $A_x = A$ or $A_x = \{0\}$. Suppose that A is ϕ-amenable for some $\phi \in \Delta(A)$. Then $\mathcal{A}$ is $(\phi \circ p_x)$-character amenable for all $x \in X$. Thus, if A is character amenable, so is $\mathcal{A}$.*

Proof. A character $H \in \Delta(\mathcal{A})$ has the form $H = \phi \circ p_x$ for some $x \in X$ and $\phi \in \Delta(A) = \Delta(A_x)$ $(A_x \neq \{0\})$. Fix x, and suppose that A is ϕ-amenable. For each y such that $A_y = A \neq 0$, choose $\phi_y = \phi \in \Delta(A_y)$. Then $\Phi = \{\phi_y = \phi \in \Delta(A_y) : A_y \neq \{0\}\}$ is a uniformly amenable collection of characters; now apply the preceding theorem to see that $\mathcal{A}$ is $H = (\phi \circ p_x)$-amenable. The latter assertion follows immediately. □

Corollary 3. *Let X be locally compact, and suppose that A is character amenable. Then $C_0(X, A)$, the space of continuous A-valued functions which vanish at infinity, is also character amenable.*

Proof. Apply the preceding corollary to the discussion in Example 2. □

We now discuss the matter of locally approximate ϕ-amenability for function algebras $\mathcal{A}$. This will follow the same route as Theorem 1, but we need some preliminary results about approximate identities.

Lemma 4. *Let A be a Banach algebra, and let $I \subset A$ be a closed two-sided ideal. Suppose that I has a (right) approximate identity, and that A/I has an identity. Then A has a (right) approximate identity.*

Proof. Let $\{u_\lambda : \lambda \in \Lambda\}$ be a right approximate identity in I, and for $c \in A$, suppose that $c + I$ is the identity in A/I. Let $\Omega = \{(F, n) : F \subset A$ is finite, $n \in \mathbb{N}\}$, and order Ω by $(F', n') > (F, n)$ if $F' \supset F$ and $n' > n$.

Let $\omega = (F, n) \in \Omega$ be given, say $F = \{a_{1,...,}a_m\}$. Then for each $a_i \in F$, we have $a_i c - a_i \in I$, and we clearly can choose $u_{\lambda_\omega} \in \{u_\lambda\}$ such that $\|(a_i c - a_i)u_{\lambda_\omega} - (a_i c - a_i)\| < 1/n$ for each $a_i \in F$. Re-ordering the summands above, we get

$$\|a_i(u_{\lambda_\omega} + c - cu_{\lambda_\omega}) - a_i\| < 1/n.$$

Then Ω indexes an approximate identity for A. For, given $a \in A$, let $F = \{a_1, ..., a_m\}$ be a finite set containing a, and let $n \in \mathbb{N}$ be given, with $\omega = (F, n)$. Then there exists $u_{\lambda_\omega} \in \{u_\lambda\}$ such that $\|a_i(u_{\lambda_\omega} + c - cu_{\lambda_\omega}) - a_i\| < 1/n$ for all $a_i \in F$. If $\omega' = (F', n') > \omega = (F, n)$, then surely

$$\left\|a(u_{\lambda_{\omega'}} + c - cu_{\lambda_{\omega'}}) - a\right\| < 1/n' < 1/n.$$

□

Lemma 5. *Let $\mathcal{A}$ be a function algebra, and suppose each A_x has a right approximate identity $\{u_{\lambda_x} : \lambda_x \in \Lambda_x\}$. Then $\mathcal{A}$ has a right approximate identity.*

Proof. We adapt the proofs of Proposition 1.2 of [2] and Lemma 4 and Corollary 3 of [6].

Let $\Lambda = \prod\{\Lambda_x : x \in X\}$, where Λ_x is the index set for the approximate identity in A_x : we have $au_{\lambda_x} \to a$ for each $a \in A_x$. Order Λ pointwise, and for greater typographic clarity write $\lambda(x) = \lambda_x \in \Lambda_x$. Let $F = \{\sigma_i : i = 1, ..., m\} \subset \mathcal{A}$ be a finite set, and let $n \in \mathbb{N}$.

Let $x \in X$. For each $u_{\lambda(x)} \in A_x$, we can choose $\nu_{\lambda(x)} \in \mathcal{A}$ such that $\nu_{\lambda(x)}(x) = u_{\lambda(x)}$ and such that $\left\|\nu_{\lambda(x)}\right\| = \left\|u_{\lambda(x)}\right\|$ (see [12, Proposition 1.1].) For each $i = 1, ..., m$, choose $\lambda_{n,i}(x) \in \Lambda_x$ such that if $\lambda(x) \geq \lambda_{n,i}(x)$ then $\left\|\sigma_i(x) - \sigma_i(x)\nu_{\lambda(x)}(x)\right\| < 1/n$. Then if $\lambda_n \in \Lambda$ is such that $\lambda_n(x) \geq \max\{\lambda_{n,i}(x) : i = 1, ..., m\}$, we have $\left\|\sigma_i(x) - \sigma_i(x)\nu_{\lambda_n(x)}(x)\right\| < 1/n$ for all $x \in X$ and $i = 1, ..., m$.

From the upper semicontinuity of the norm function in $\mathcal{A}$, for each $x \in X$ we can then choose an open neighborhood $V_x(F, n)$ of x such that if $y \in V_x(F, n)$, then $\left\|\sigma_i(y) - \sigma_i(y)\nu_{\lambda_n(x)}(y)\right\| < 1/n$ for all $\sigma_i \in F$. Since X is compact, we can choose $\{x_j : j = 1, ..., s\} \subset X$ such that $\{V_j\} = \{V_{x_j}(F, n), j = 1, ..., s\}$ also covers X. Let $\{f_j : j = 1, ..., s\}$ be a partition of unity subordinate to the V_j; i.e. each $f_j : X \to [0, 1]$ is continuous and supported on V_j, and $\sum f_j = 1$. Define $\xi = \xi(F, n)$ by $\xi = \sum f_j \nu_{\lambda_n(x_j)}$. Then, for $y \in X$ and $\sigma_i \in F$, we have

$$\begin{aligned}
\|\sigma_i(y) - \sigma_i(y)\xi(y)\| &= \left\|\textstyle\sum_j f_j(y)[\sigma_i(y) - \sigma_i(y)\nu_{\lambda_n(x)}(y)]\right\| \\
&\leq \sum_{j \text{ s.t. } y \in V_j} f_j(y) \left\|\sigma_i(y) - \sigma_i(y)\nu_{\lambda_n(x)}(y)\right\| \\
&\leq \sum_{j \text{ s.t. } y \in V_j} f_j(y) * 1/n \\
&< 1/n.
\end{aligned}$$

Thus, $\|\sigma_i - \sigma_i\xi\| < 1/n$ for each $\sigma_i \in F$.

Now, set $\Omega = \{(F, n) : F \subset \mathcal{A} \text{ is finite}, n \in \mathbb{N}\}$, and order Ω by $(F', n') > (F, n)$ if $F' \supset F$ and $n' > n$. Then, proceed as in the proof of Lemma 4. Let $\sigma \in \mathcal{A}$, and for $\omega = (F, n) \in \Omega$, let $F = \{\sigma_1, ..., \sigma_m\} \subset \mathcal{A}$ be a finite set containing σ. Then there exists $\xi_\omega = \xi(\omega) \in \mathcal{A}$ such that $\|\sigma_i - \sigma_i\xi_\omega\| < 1/n$, for all $\sigma_i \in F$. Hence, if $\omega' = (F', n') > \omega = (F, n)$, we have $\|\sigma - \sigma\xi_{\omega'}\| < 1/n' < 1/n$. □

Theorem 2. *Suppose that $\mathcal{A}$ is a function algebra over X, and there is a collection $\Phi = \{\phi_y \in \Delta(A_y) : A_y \neq \{0\}\}$ of characters such that each A_y is locally approximately ϕ_y-amenable. If A_x is locally approximately ψ_x-amenable for some $\psi_x \in \Delta(A_x)$, then $\mathcal{A}$ is locally approximately $H_{\psi_x} = \psi_x \circ p_x$-amenable.*

Proof. It suffices to exhibit a right approximate identity for $\mathcal{K}_{\psi_x} = \{\sigma \in \mathcal{A} : \sigma(x) \in \ker \psi_x\}$. From Lemma 4, the A_y $(y \neq x)$ have approximate identities since, for any $u_y \in A_y \neq \{0\}$ such that $\phi_y(u_y) = 1$, $u_y + \ker \phi_y$ is the identity in $A_y / \ker \phi_y$. (Such u_y exist because $0 \neq \phi_y \in A_y^*$.) If we replace $\phi_x \in \Phi$ with ψ_x, Lemma 5 assures us that $\mathcal{K}_{\psi_x}$ has an approximate identity, so that $\mathcal{A}$ is locally approximately ψ_x-amenable. □

The same corollaries as above can then be generated. Probably the most interesting is the following.

Corollary 4. *Let A be a Banach algebra, and X a locally compact space. If $\phi \in \Delta(A)$, and A is locally approximately ϕ-amenable, then $\mathcal{A} = C_0(X, A)$ is locally approximately $H = \phi \circ p_x$-amenable.*

Compare Theorems 1 and 2 to Theorem 2.2 of [14]. The proof of the latter employs the existence of locally constant functions in $C_0(X, A)$. In our situation, however, the fibers of $\mathcal{A}$ need not be constant nor, even if they are constant, must $\mathcal{A}$ contain locally constant functions. Lacking this, however, we instead are able to use the workaround of (bounded) approximate identities in various spaces.

The following lemma and example show one way in which it is possible to guarantee that, even though each fiber of a function algebra $\mathcal{A}$ is character amenable, $\mathcal{A}$ itself is not ϕ-amenable for any $\phi \in \Delta(\mathcal{A})$.

Lemma 6. *Let A be a Banach algebra with (right) approximate identity $\{u_\gamma : \gamma \in \Gamma\}$ bounded by m, and let $\phi \in \Delta(A)$. Then $\|\phi\| \leq 1/m$.*

Proof. It is easy to check that $\phi(u_\gamma) \to 1$. Hence, for given $\delta, 0 < \delta < 1$, there exists $u \in \{u_\gamma\}$ such that $||\phi(u)| - 1| < \delta$, or $|\phi(u)| > 1 - \delta$. Thus,

$$\|\phi\| = \sup\{|\phi(a)| : \|a\| \leq 1, a \in A\} \geq |\phi(u)| / m > (1 - \delta)/m.$$

Since δ was arbitrary, we have $\|\phi\| \geq 1/m$. □

If follows that if $\inf\{\|\phi\| : \phi \in \Delta(A)\} = 0$, then A has no bounded (right) approximate identity, and hence that A is not ϕ-amenable for any $\phi \in \Delta(A)$.

Using the above lemma, we can then work through the details of a simple example which shows that a function algebra $\mathcal{A}$ (with fibers A_x over $x \in X$) may be such that each A_x is character amenable, yet $\mathcal{A}$ is only locally approximately $H_{\phi_x} = \phi_x \circ p_x$-amenable for each $H_{\phi_x} \in \Delta(\mathcal{A})$.

Example 3. Let $X = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$, and set $A_n = \mathbb{C}$ if $n \neq \infty$, $A_\infty = \{0\}$. If $\alpha \in A_n$, set $\|\alpha\|_n = n\,|\alpha|$, and let $\mathcal{A}$ be the set of functions σ on X such that $\sigma(n) \in A_n$ and $\|\sigma(n)\|_n \to 0$. Then $\Delta(A_n) = \{i_n\}$, where i_n is the identity map on A_n if $n \neq \infty$, $\Delta(A_\infty) = \varnothing$, $\|i_n\| = 1/n$, and $\mathcal{K}_{i_n} = \{\sigma \in \mathcal{A} : \sigma(n) = 0\}$. Evidently, A_n has identity $1, \|1\|_n = n$, and A_n is i_n-amenable, and hence character amenable. However, the ideal $\mathcal{K}_{i_n}$ has only an unbounded (right) approximate identity. (Proof: Define $\tau_n \in \mathcal{K}_{i_n}$ by $\tau_n(k) = 1$ if $k = n+1$, $\tau_n(k) = 0$ otherwise. If $\sigma_\lambda \in \mathcal{K}_{i_n}$ is a (right) approximate identity for $\mathcal{K}_{i_n}$, then $\lim_\lambda \tau_n\sigma_\lambda = \tau_n$ for each n; in particular, $\tau_n(n+1)\sigma_\lambda(n+1) \xrightarrow{\lambda} \tau(n+1) = 1$. So $\sigma_\lambda(n+1) \to 1 \in A_{n+1}$, and since $\|1\|_{n+1} = n+1$, it follows that $\limsup \|\sigma_\lambda\| \geq n+1$, so that $\{\sigma_\lambda\}$ is unbounded.) (See a similar discussion in [13, Example 19].) Hence, while $\mathcal{A}$ is locally approximately $H_n = i_n \circ p_n$-amenable for each H_n, it is not H_n-amenable for any H_n.

Finally, we note the following for completeness.

Proposition 4. *Let $\mathcal{A}$ be a function algebra over the compact Hausdorff space X. Let $x \in X$, set $H_{\phi_x} = \phi_x \circ p_x$, and suppose that $\mathcal{A}$ is (locally approximately) H_{ϕ_x}-amenable. Then A_x is (locally approximately) ϕ_x-amenable. Consequently, if $\mathcal{A}$ is character amenable, then each fiber A_x is character amenable.*

Proof. Let $\{\sigma_\lambda : \lambda \in \Lambda\}$ be a (bounded) right approximate identity for $\ker H_{\phi_x} = \mathcal{K}_{\phi_x}$. If $a \in \ker \phi_x \subset A_x$, we can choose $\tau \in \mathcal{K}_{\phi_x}$ such that $\tau(x) = a$. Then, for any such $\tau \in \ker H_{\phi_x}$, we have $\|\tau\sigma_\lambda - \tau\| \to 0$; since the evaluation map p_x is norm decreasing, we then also have $\|a\sigma_\lambda(x) - a\| \to 0$. Evidently, if $\{\sigma_\lambda\}$ is bounded, then so is $\{\sigma_\lambda(x)\}$, so that a bounded right approximate identity for $\mathcal{K}_{\phi_x}$ forces the same for $\ker \phi_x$. If $\{\sigma_\lambda\}$ is unbounded, then at worst we have an unbounded approximate identity for $\ker \phi_x$. □

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Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458 USA

E-mail address: terjehill@fau.edu

5012 N. Via Velazquez, Tucson, AZ 85750 USA; Trinity College, Hartford, CT 06106 USA

E-mail address: david.robbins@trincoll.edu