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Quantum test of the distributions of composite physical measurements

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Abstract

Probability distribution functions and lowest statistical moments of composite measurements representable as products and quotients of independent normal variates are derived, and tested by means of the α and β branching decays of ^{212}Bi. The exact composite distribution functions are nongaussian and provide correct uncertainty estimates and confidence intervals in cases where standard error propagation relations are inaccurate. Although nuclear decay processes give rise to Poisson-distributed parent populations, the gaussian-based composite distributions form nearly perfect envelopes to the discrete distributions of products and ratios of Poisson variates, even for relatively low counts. To our knowledge, this is the first reported experimental test of the statistics of composite measurements by a fundamental quantum process.

Running Head: Quantum test of distributions

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Introduction

Many, if not most, experimental quantities of interest in physics, engineering, and medicine are not measured directly, but are inferred from products and ratios of measurements. Individual measurements entering into composite quantities may be regarded as random variables for which the probability density function (pdf), cumulative probability function (cpf), mean, variance, and other statistics are usually known or ascertainable. However, the distributions of the composite measurements are in general different from those of the parent distributions, a circumstance that pertains to products and ratios of normal variates, which is one of the most widely occurring cases in the physical and life sciences.

Although standard error-propagation formulas for determining uncertainties of measurements without knowledge of the exact pdf or cpf are in wide use [1] [2], these rules may be inadequate as they are approximations derived from Taylor-series expansions that can fail entirely for certain parent distributions. Moreover, without knowledge of the pdf or cpf, one cannot rigorously assign confidence intervals to the uncertainties derived from these rules. Measurements reported in the physics literature as a mean value μ plus or minus a standard deviation σ are ordinarily assumed to distributed normally with 68% of the area under the pdf symmetrically distributed within 2σ about μ. This assumption can be false, however, for nongaussian distributions, particularly those with pronounced skewness.

The central limit theorem (CLT) establishes conditions under which sums of independent random variables (and therefore the mean of a series of measurements) are asymptotically normally distributed [3]. The CLT, however, says nothing about the rate of convergence and is inapplicable in many practical cases where the number of measurements taken are too few. Important cases of this kind abound in nuclear and elementary particle physics in the study of rare processes leading to low signal counts [4], or in clinical medicine where numerous indices (such as the ratio of total cholesterol to high-density lipoprotein cholesterol) are routinely determined for each patient by a single trial on a single tissue sample performed at most once per year [5]. There are also instances where the CLT is intrinsically inapplicable regardless of the number of measurements, as in the case of a Cauchy distribution (which arises from the ratio of two standard normal variates).[6]

Although statisticians have examined some theoretical aspects of the ratios of normal variates [7][8][9], standard monographs and current reviews of statistical methodology widely used in physics do not even mention the subject.[10][11][12][13], and, besides our preliminary study [14], we are aware of no reports in the physics literature in which the statistical theory of composite measurements has been rigorously tested or employed.

In this paper we derive expressions for the cpf, pdf, and associated moments of both the product and ratio of two independent normal variates, $X = N(\mu_1, \sigma_1^2)$ and $Y = N(\mu_2, \sigma_2^2)$, where the designation $N(\mu, \sigma^2)$ implies a pdf of the form $p(x) = \left(2\pi\sigma^2\right)^{-1/2} e^{-(x-\mu)^2/2\sigma^2}$ with parameters μ (mean) and σ (standard deviation). Next, we test the theory experimentally by constructing product and quotient distributions from measurements of the branching α and β decays of radioactive ^{212}Bi under conditions where the Poisson-distributed parent samples are well represented by normal distributions. Last, we discuss briefly physical implications of parent distributions other than gaussian, in particular Poisson and gamma distributions.

Theory

Consider first the density $p_Z(z)$ of $Z = XY$ (where we follow the convention of representing a random variable by an upper-case letter and its realisation by the corresponding lower-case letter). The cumulative distribution $P(z) \equiv P(xy \le z)$ is given in terms of the conditional probability $P(x \le (z/y)|y)$ and pdf $p_Y(y)$ by $P(z) = \int_{-\infty}^{\infty} P(x \le (z/y)|y)\, p_Y(y)dy$ in which $P(x \le (z/y)|y) = \int^{z/y} p_X(x)dx$. The product pdf, $p_Z(z) = dP(z)/dz$, then takes the form

$p_Z(z) = \int p_X(y/z)\, p_Y(y) |y|^{-1} dy$, which, upon substitution of the gaussian pdfs for X and Y and transformation of the integration variable, leads to

$$p_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2}\int_0^{\infty}\left[\exp\left\{-\left(\frac{\left(ze^{-w}-\mu_1\right)^2}{2\sigma_1^2}+\frac{\left(e^{w}-\mu_2\right)^2}{2\sigma_2^2}\right)\right\}+\exp\left\{-\left(\frac{\left(ze^{-w}+\mu_1\right)^2}{2\sigma_1^2}+\frac{\left(e^{w}+\mu_2\right)^2}{2\sigma_2^2}\right)\right\}\right]dw \tag{1a}$$

If the domains of X and Y are restricted to positive values, as corresponds to many physical applications, Eq. (1a) reduces to

$$p_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-(\mu_2/\sigma_2)}^{\infty} e^{-w^2/2}\exp\left\{-\frac{\left(z-\mu_1\mu_2-\mu_1\sigma_2 w\right)^2}{2\sigma_1^2\sigma_2^2\left(\frac{\mu_2}{\sigma_2}+w\right)^2}\right\}\left(\frac{\mu_2}{\sigma_2}+w\right)^{-1} dw. \tag{1b}$$

Under the conditions $\mu_i/\sigma_i >> 1$ $(i=1,2)$, Eq. (1b) reduces to the normal distribution $N(\mu_z, \sigma_z^{o2})$ with $\mu_z = \mu_1\mu_2$ and $\sigma_z^{o2} = \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2$, which are the results of standard error propagation formulas. However, in general pdf (1a) differs markedly from a gaussian. For example, in the special case $X = N_1(0,1)$, $Y = N_2(0,1)$, where the variables span the full set of real numbers, pdf (1a) yields a cusped density function with standard deviation $\sigma_z = 1$ (not $\sigma_z^o = 0$).

The moments of the product distribution can be evaluated exactly from the pdf of the factors

$$m_n \equiv \int z^n p_z(z)dz = \int x^n p_X(x)dx \int y^n p_Y(y)dy \equiv m_n^{(x)} m_n^{(y)}. \tag{2a}$$

where

$$m_n^{(i)} = \mu_i^n \sum_0^n \binom{n}{k}\left(\frac{\sigma_i}{\mu_i}\right)^k \times \begin{cases} 1 & k = 0 \\ 0 & \text{odd k} \\ (k-1)!! & \text{even k} \geq 2 \end{cases}. \tag{2b}$$

The first three moments (a) $m_1 = \mu_1\mu_2$, (b) $m_2 = \left(\mu_1^2 + \sigma_1^2\right)\left(\mu_2^2 + \sigma_2^2\right)$, and (c) $m_3 = \mu_1\mu_2\left(\mu_1^2 + 3\sigma_1^2\right)\left(\mu_2^2 + 3\sigma_2^2\right)$ lead to the variance

$$V \equiv m_2 - m_1^2 = \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2 \tag{3a}$$

and skewness

$$Sk \equiv \left(\sigma_z^3\right)^{-1}\int (z-m_1)^3\, p_Z(z)dz = 6\mu_1\mu_2\sigma_1^2\sigma_2^2\left(\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2\right)^{-3/2} \tag{3b}$$

Skewness is a measure of the asymmetry of a probability distribution about the mean, and although $Z = XY$ is symmetric in its factors, each of which is distributed according to a symmetric pdf, the product distribution itself has in general a nonvanishing skewness.

Consider next the density $p_Z(z)$ of $Z = X/Y$ where the realisations of X and Y can span the full set of real numbers. By taking the derivative $p_Z(z) = dP(z)/dz$ of the cumulative distribution $P(z) \equiv P(x/y \le z) = \int_{-\infty}^{\infty} P\left(x \le (yz)\middle| y\right) p_Y(y)dy$, we obtain the quotient pdf $p_Z(z) = \int p_X(yz)\, p_Y(y)|y|dy$, which, for normally distributed X and Y, evaluates to

$$\begin{aligned} p_z(z) = &\frac{1}{\sqrt{2\pi}}\left(\frac{\mu_1\sigma_2^2 z + \mu_2\sigma_1^2}{\left(\sigma_2^2 z^2 + \sigma_1^2\right)^{3/2}}\right)\exp\left\{-\left(\frac{(\mu_1 - \mu_2 z)^2}{2\left(\sigma_2^2 z^2 + \sigma_1^2\right)}\right)\right\}\operatorname{erf}\left(\frac{\mu_1\sigma_2^2 z + \mu_2\sigma_1^2}{\sqrt{2}\sigma_1\sigma_2\left(\sigma_2^2 z^2 + \sigma_1^2\right)^{1/2}}\right) \\ &+ \frac{1}{\pi}\left(\frac{\sigma_1\sigma_2}{\sigma_2^2 z^2 + \sigma_1^2}\right)\exp\left\{-\left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2}\right)\right\} \end{aligned} \tag{4a}$$

Under the conditions that X and Y be nonnegative with $\mu_i/\sigma_i >> 1$ $(i = 1,2)$, Eq. (4a) can be approximated by the non gaussian expression

$$p_z(z) = \frac{1}{\sqrt{2\pi}}\left(\frac{\mu_1\sigma_2^2 z + \mu_2\sigma_1^2}{\left(\sigma_2^2 z^2 + \sigma_1^2\right)^{3/2}}\right)\exp\left\{-\left(\frac{(\mu_1 - \mu_2 z)^2}{2\left(\sigma_2^2 z + \sigma_1^2\right)}\right)\right\}, \tag{4b}$$

Replacement of z by the measured value μ_1/μ_2 in the prefactor and denominator of the exponential in Eq. (4b) reduces $p_Z(z)$ to the normal distribution $N(\mu_z, \sigma_z^{o2})$ with $\mu_z = \mu_1/\mu_2$ and $\sigma_z^{o2} = \sigma_1^2\mu_2^{-2} + \mu_1^2\sigma_2^2\mu_2^{-4}$. The latter expression is again a common error propagation formula, but, Eq. (4a) leads in general to distributions markedly different from normal.

The moments of Z,

$$m_n \equiv \int z^n p_z(z)dz = \int x^n p_X(x)dx \int y^{-n} p_Y(y)dy\ , \tag{5}$$

if not obtained by direct numerical integration, can be approximated in closed-form algebraic expressions by performing a series expansion in σ_2 / μ_2 of the integral in Eq. (5) corresponding to the expectation value $E(Y^{-n})$, which may be cast in the form

$$E(Y^{-n}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-w^2/2}\left[(\mu_2 + \sigma_2 w)^{-n} + (\mu_2 - \sigma_2 w)^{-n}\right] dw - \frac{1}{\sqrt{2\pi}} \int_{\mu_2/\sigma_2}^\infty e^{-w^2/2}(\mu_2 - \sigma_2 w)^{-n}\, dw \tag{6a}$$

and reduces to

$$E(Y^{-n}) = \frac{1}{\mu_2^n} \sum_{r=0}^{r_{max}} \left(\frac{\sigma_2}{\mu_2}\right)^{2r} \frac{(n+2r-1)!}{2^r r!(n-1)!}. \tag{6b}$$

The value of $r_{\max}$ in Eq. (6b) is determined by the convergence criterion that the $(r+1)^{th}$ term be smaller than the r^{th} term, which leads to the inequality $(\sigma_2 / \mu_2)^2 < 2(r+1)[(n+2r)(n+2r+1)]^{-1}$. Substitution of Eq. (6b) into Eq. (5) results in the following moments, truncated at order $(\sigma_2 / \mu_2)^8$:

$$m_1 \approx (\mu_1 / \mu_2)\left[1 + (\sigma_2/\mu_2)^2 + 3(\sigma_2/\mu_2)^4 + 15(\sigma_2/\mu_2)^6 + 105(\sigma_2/\mu_2)^8 \cdots\right] \tag{7a}$$

$$m_2 \approx \left((\mu_1^2 + \sigma_1^2)/\mu_2^2\right)\left[1 + 3(\sigma_2/\mu_2)^2 + 15(\sigma_2/\mu_2)^4 + 105(\sigma_2/\mu_2)^6 + 945(\sigma_2/\mu_2)^8 \cdots\right] \tag{7b}$$

$$m_3 \approx \left(\mu_1\left(\mu_1^2 + 3\sigma_1^2\right)/\mu_2^3\right)\left[1 + 6(\sigma_2/\mu_2)^2 + 45(\sigma_2/\mu_2)^4 + 420(\sigma_2/\mu_2)^6 + 4725(\sigma_2/\mu_2)^8 \cdots\right] \tag{7c}$$

In general, the distribution of $Z = X/Y$ departs much more markedly than the distribution of $Z = XY$ from a normal distribution, as will be illustrated below.

Generation of Composite Distributions by Nuclear Decay

To test the relations in the preceding section, we determined the distributions of the product and ratio of radioactive decays via the two branching decay modes of ^{212}Bi:

(a) $^{212}Bi \rightarrow {}^{212}Po + \beta$ (β branch ratio 64.06%)

(b) $^{212}Bi \rightarrow {}^{208}Tl + \alpha$ (α branch ratio 35.94%)

The transmutation of atomic nuclei is a quantum process, which, as demonstrated quantitatively in our recent experimental tests [15], occurs randomly and without regard to past influences; it is, as far as is known, nature's most perfect random number generator.

Our procedure, in brief, is as follows. A 250 g sample of 40 year old powdered thorium dioxide provided a source of ^{224}Ra in secular equilibrium. The source was placed in a sealed aluminium chamber in which a silicon surface barrier detector had been mounted. A potential of -1 kV with respect to the grounded chamber was applied to the detector, allowing electrostatic precipitation of ionized ^{224}Ra progeny, particularly ^{216}Po. The precipitation proceeded until a suitable level of activity was achieved. The detector was then removed from the chamber and connected to standard nuclear electronics for α pulse height analysis (PHA) and multichannel scaling (MCS). ^{216}Po decays by alpha emission to ^{212}Pb which, in turn, decays to ^{212}Bi, the nuclide of interest. There are two decay pathways: an alpha mode (36% branch ratio) resulting in ^{208}Tl with emission of a 6.05 Mev or 6.09 MeV alpha particle; a beta mode (64% branch ratio) resulting in ^{212}Po with a halflife of 0.3 μs, which promptly decays to ^{208}Pb with emission of an 8.78 MeV alpha particle.

We generated data using PHA α spectroscopy [16] to isolate the peaks produced by ^{212}Bi and ^{212}Po, and record the number of events per time interval via MCS. The energy resolution of the spectrometer was approximately 15 keV per bin, giving a separation of peaks of about 180 bins with peak widths about 30 bins. Thus, the two peaks were well isolated and the rates of each branch determined. The measured rate was about 150 events per second for the Po branch and approximately 100 for the Bi branch. The dwell time was 0.05 s per bin for the 8196 bin samplings. The duration of each run was 409.6 s and the runs were taken sequentially with an interval of approximately 3 minutes between to allow adjustment of the energy window.

By combining the contents of 2, 4, 8, etc. contiguous bins, we derived from the same data set parent populations of increasing mean counts for each decay mode. One element of corresponding type (e.g. a 2-bin or 8-bin count) from each of the two decay modes was then chosen to form a product and quotient. Although the parent populations follow a Poisson distribution, the respective means can be made sufficiently large that the envelope of each parent Poisson distribution is well approximated by the corresponding normal distribution $N(\mu_i,\mu_i)$ $(i=1,2)$ as shown in Figure 1 for $\mu_1=\mu_{Po}=14.7$, $\mu_2=\mu_{Bi}=9.8$, resulting from 2-bin combination. Figure 2 shows the quotient distribution for data sets derived from 2-bin combinations. The visual fit of the theoretical distribution function as an envelope to the experimental histogram is excellent and leads to reliable values for the means, widths, and asymmetries of the quotient distributions, as summarized in Table 1, which compares sample statistics to corresponding statistics calculated from the pdf of Eq. (4a). Excellent agreement is also obtained for the product distribution of the two decay modes, as illustrated in Figure 3 for the 2-bin data, with moments also summarized in Table 1.

Table 1: Experimental and Theoretical Moments for $\mu_1=14.7$, $\mu_2=9.8$

	$^{212}Po(\mu_1)/^{212}Bi(\mu_2)$		$^{212}Po(\mu_1)\times{}^{212}Bi(\mu_2)$	
	Sample	PDF	Sample	PDF
m1	1.683	1.638	145.322	145.307
m2	3.660	3.520	$2.475\ 10^4$	$2.483\ 10^4$
m3	12.744	12.349	$4.820\ 10^6$	$3.608\ 10^6$
$\sigma=\sqrt{V}$	0.909	0.914	60.264	60.986
Sk	5.069	5.033	0.769	0.559

Agreement between theory and experiment was obtained for all n-bin parent populations, $2\le n\le 64$.

A chi-square test of goodness of fit is not really pertinent here, as we know at the outset the data derive from discrete rather than continuous parent distributions. Nevertheless, the results of a chi-square test are interesting. Analysis of the quotient 8-bin data yields a probability $P(\chi^2>\chi^2_{obs})$

= 24.3%, where χ^2_{obs} is the observed value for d = 14 degrees of freedom, indicating that rejection of the theoretical distribution as a fit to the data would be unwarranted. However, $P(\chi^2 > \chi^2_{obs})$ is close to 0% for the 2-bin data despite the excellent visual match and moment predictions. This curious feature is due to the fact that the true Poisson ratio distribution function

$$f_{Z=X/Y}(z) = e^{-(\mu_1+\mu_2)} \sum_{\substack{y=1 \\ [yz=\text{integer}]}}^{\infty} \frac{\mu_2^{y}\mu_1^{yz}}{y!(yz)!}\,, \tag{8a}$$

which we derived by reasoning analogous to that leading to pdf (4a), generates rational numbers rather than the set of all real numbers, with the consequence that $f_Z(z)$ exhibits naturally occurring "lacunae", i.e. values of z at which $f_Z(z)$ drops suddenly to zero or close to zero. These appear as fluctuations in Figure 2, but they are fully reproducible, as shown in Figure 4, and do not vanish with increasing sample size. The influence of discreteness diminishes, however, as the mean parameters of the parent distributions increase, which is the condition under which a Poisson distribution tends toward a normal distribution. Similar pseudo fluctuations are exhibited by the true Poisson product distribution function

$$f_{Z=XY}(z) = e^{-(\mu_1+\mu_2)} \sum_{\substack{y=1 \\ [z/y=\text{integer}]}}^{\infty} \frac{\mu_2^{y}\mu_1^{y/z}}{y!(z/y)!}\,. \tag{8b}$$

Conclusion: Some Cautionary Remarks

We have deduced and tested experimentally the distributions and lowest three moments of the product and ratio of two normally distributed independent, uncorrelated, direct measurements. From our relations, the distribution of composite measurements comprising any combination of products and ratios of direct measurements can be calculated. (We consider elsewhere measurements involving powers higher than unity.) For example, the pdf $p_W(w)$ of $W = XY/Z$, which might represent measurement of the gas constant $R = PV/T$ from measurements of pressure, molar volume, and absolute temperature, is given by

$$p_W(w) = \iint p_Z(z)\,p_{XY}(wz)|z|dz = \int p_Z(z)|z|dz \int p_Y(y)\,p_X(\tfrac{wz}{y})|y|^{-1}dy \tag{9}$$

in terms of the pdfs of the component measurements. In experiments for which there is no *a priori* reason to expect a composite measurement to be distributed normally, it would be prudent, if not necessary, to use the distributions tested here, rather than resorting to approximate error propagation formulas.

In general, our theory predicts, and the nuclear decay experiments support, the result that the product and quotient distributions of normal variates, although not themselves normal, become increasingly better approximated by normal distributions as the ratio of mean to standard deviation of the parent distributions increases. It may be thought that this characteristic feature applies to *arbitrary* parent distributions, but here one must exercise caution. For example, for a random variable governed by a gamma distribution $p_{m,\theta}(u) = \frac{\theta^{m+1}}{\Gamma(m+1)} u^m e^{-\theta u}$, the mean and standard deviation are respectively $\mu = (m+1)/\theta$ and $\sigma = \sqrt{m+1}/\theta$, leading to a ratio $\mu/\sigma = \sqrt{m+1}$ that is independent of the parameter θ. The ratio μ/σ increases with the index m, however, and in the limit of large m the exact quotient pdf, $p_Z(z) = \frac{a^{m+1} b^{m+1}}{B(m+1,n+1)} \frac{z^m}{(az+b)^{m+n+2}}$, where $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ is the beta function, is well represented by a normal distribution of corresponding mean and standard deviation. However, such a limit would be irrelevant to the study of a particular physical phenomenon whose law entails a *specifiic* value of m, as in the case of Planck's radiation law in the high-frequency domain, which takes the form of a gamma distribution with fixed index $m = 3$.

Figure Captions

Figure 1 Comparison of experimental (Poisson) parent ^{212}Bi and ^{212}Po distributions and corresponding normal distributions. Number of samples = 4096 in each parent distribution.

Figure 2 Experimental (2-bin data) quotient distribution of $\dfrac{N_{\mathrm{Po}}(14.7,14.7)}{N_{\mathrm{Bi}}(9.8,9.8)}$. Number of samples = 4096.

Figure 3 Experimental (2-bin data) product distribution of $N_{\mathrm{Po}}(14.7,14.7) \times N_{\mathrm{Bi}}(9.8,9.8)$. Number of samples = 4096.

Figure 4 Exact theoretical distribution [Eq. (8)] of Poisson variables $P(14.7)/P(9.8)$ corresponding to $^{212}\mathrm{Po}/^{212}\mathrm{Bi}$ of Figure 2.

Figure 1

Parent ^{212}Po and ^{212}Bi Distributions

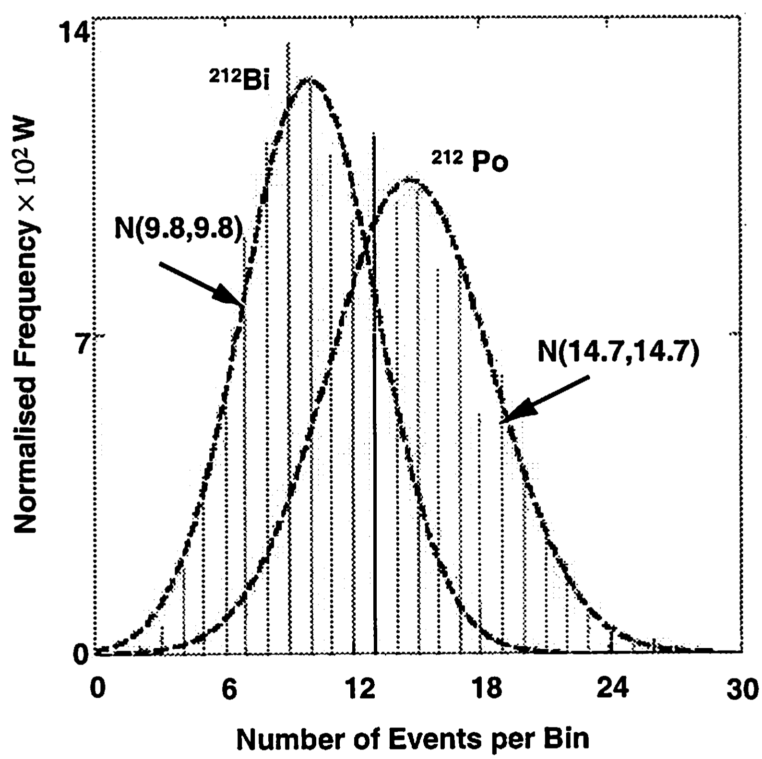


Figure 2

Distribution of $^{212}Po/^{212}Bi$ Decays

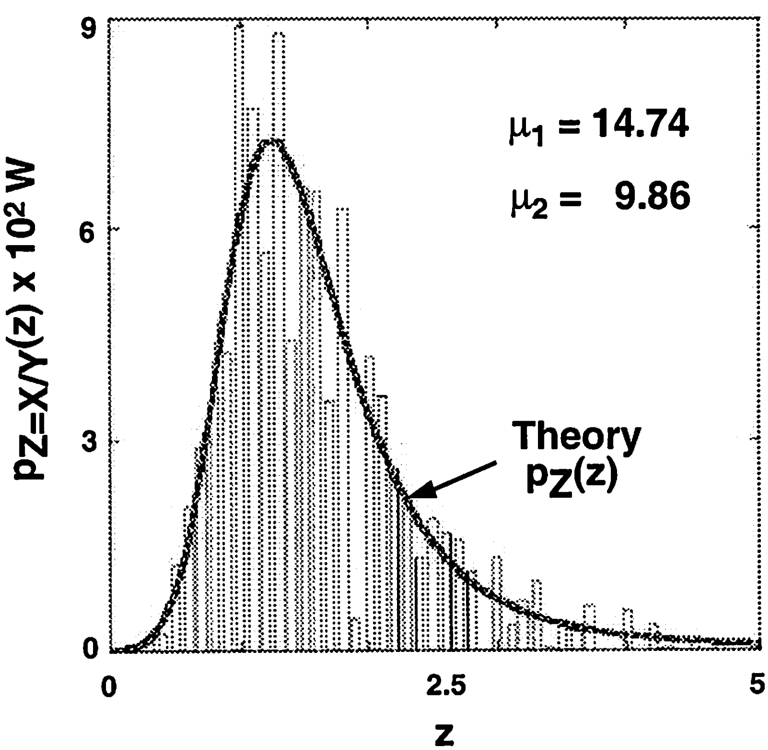


Figure 3

Distribution of ^{212}Po × ^{212}Bi Decays

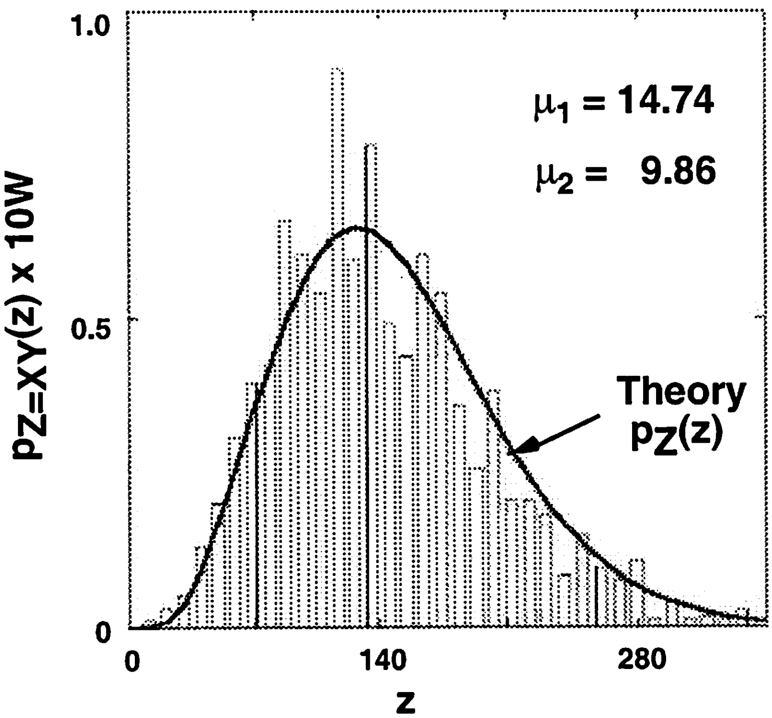
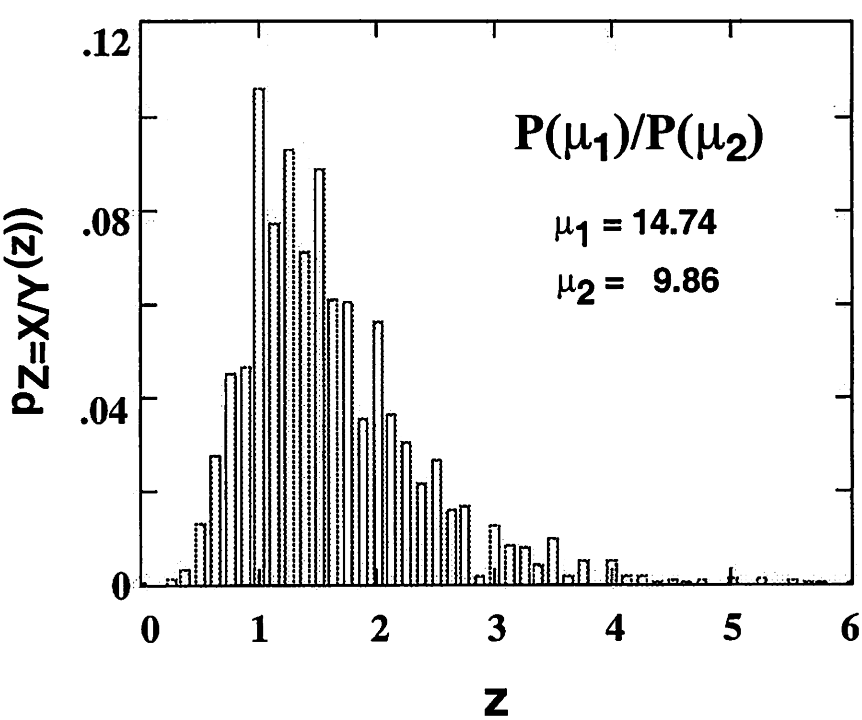


Figure 4

Theoretical Distribution of Ratio $P(\mu_1)/P(\mu_2)$



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