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Quantum Condensates in Extreme Gravity: Implications for Cold Stars and Dark Matter*

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SUMMARY

Stable end-point stars currently fall into two distinct classes—white dwarfs and neutron stars—differing enormously in central density and radial size. No stable cold dead stars are thought to span the intervening densities nor have masses beyond ~2-3 solar masses. I show, however, that the general relativistic condition of hydrostatic equilibrium augmented by the equation of state of a neutron condensate at 0 K generates stable sequences of cold stars that span the density gap and can have masses well beyond prevailing limits. The radial sizes and mass limit of each sequence are determined by the mass and scattering length of the composite bosons. Solutions for hypothetical bosons of ultra-small mass and large scattering length yield huge self-gravitating systems of low density, resembling galactic dark matter halos.

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One of the most challenging problems in contemporary astrophysics is to understand the final fate of stars that have exhausted their nuclear fuel and can no longer sustain themselves against gravitational collapse. Within the framework of Einstein's general theory of relativity, the problem is attacked by adopting an equation of state (EOS) relating the pressure and density of stellar matter and solving the Tolman-Oppenheimer-Volkoff (TOV) equations of hydrostatic equilibrium.[1] Depending on the central mass density, this procedure has led to two basic sequences of terminal equilibrium states: white dwarfs (WD) and neutron (or neutron-quark) stars (NS).

White dwarfs originate from progenitor stars of less than about five solar masses ($M_S = 2 \times 10^{30}$ kg) and are sustained by electron degeneracy pressure at radii of a few thousand kilometers with core densities of $\sim 10^9 - 10^{11}$ kg/m^3. More massive progenitors can collapse to neutron stars of a few tens of kilometers with core densities $\sim 10^{17} - 10^{19}$ kg/m^3 maintained by the degeneracy pressure of unbound neutrons formed by inverse beta decay of disrupted atomic nuclei. No stable states are predicted within the wide gap of densities between the WD and NS regions nor in the limitless range beyond NS densities. Moreover, the TOV solutions have led to mass limits of about 2-3 M_S for both white dwarfs and neutron stars. Cold stars over the limit collapse to black holes where matter falls into a central singularity and the laws of physics break down.

In this paper I solve the TOV equations for a self-gravitating Bose-Einstein condensate (BEC) at 0 K formed by fermion condensation of neutrons into composite bosons. In contrast to previous results for unbound neutronic matter, I show that it may be possible for stable equilibrium states of neutron BEC stars to exist in the range between WD and NS core densities and have masses much higher than the Chandrasekhar (for WD) or Oppenheimer-Volkoff (for NS) limits.

The energy density ε_b of a quantum condensate is derivable from the Gross-Pitaevskii (GP) nonlinear Schrödinger equation[2]. Under steady-state conditions with neglect of kinetic energy (Thomas-Fermi approximation valid at 0 K) and addition of a rest-mass contribution, the energy density ε_b takes the dimensionless form

$$x_b \equiv \frac{\varepsilon_b}{\left(m_b c^2/v\right)} = \eta_b + \tfrac{1}{2}(\eta_b)^2 + \tfrac{1}{8}\lambda\!\!\!^-{}_b^2 \frac{(\nabla\eta_b)^2}{\eta_b} \tag{1a}$$

where m_b is the boson mass, n_b is the boson number density, and $\eta_b = n_b v$ is the number of particles in an interaction volume v defined by

$$v = 4\pi a \lambda\!\!\!^-{}_b^2 \tag{1b}$$

in which $\lambda\!\!\!^-{}_b = \hbar / m_b c$ is the (reduced) boson Compton wavelength and a is the scattering length that appears in the GP equation. The last term in Eq. (1a) is a quantum zero-point energy which can be neglected when the condensate healing length $\xi_b = (8\pi a n_b)^{-1/2}$ is much smaller than the mean particle separation $d_b = n_b^{-1/3}$ as will be shown to be the case for condensed neutrons in a neutron star. Throughout the paper I retain familiar units employing the speed of light c, universal gravitational constant G, and (reduced) Planck's constant $\hbar = h/2\pi$.

From Eq. (1a) follow dimensionless forms for the boson chemical potential μ_b

$$\zeta_b \equiv \frac{\mu_b}{\left(m_b c^2\right)} = \frac{dx_b}{d\eta_b} = 1 + \eta_b \tag{2}$$

and condensate pressure P_b

$$z_b \equiv \frac{P_b}{\left(m_b c^2/v\right)} = \eta_b\zeta_b - x_b = \tfrac{1}{2}\eta_b{}^2, \tag{3}$$

from which one derives the condensate EOS in a form suitable for the TOV equation

Pressure-Energy $\quad x_b = z_b + \sqrt{2z_b} \quad$ or $\quad z_b = \tfrac{1}{2}\left[\sqrt{1+2x_b} - 1\right]^2 \quad$ (4a)

Pressure-Chemical Potential $$z_b = \tfrac{1}{2}(\zeta_b - 1)^2 \,. \tag{4b}$$

It is to be noted that EOS (4a,b) is causal; i.e. the derivative,

$$\frac{dP_b}{d\varepsilon_b} = \frac{dz_b}{dx_b} = 1 - \zeta_b^{-1} , \tag{5}$$

which corresponds to the square of the ratio of the speed of sound to the speed of light, is always in the range $1 > \frac{dP_b}{d\varepsilon_b} \geq 0$ because ζ_b is always ≥ 1.

Hydrostatic equilibrium within a nonrotating neutron-condensate star at 0 K in a Schwarzschild geometry is expressed by the set of TOV equations in the form

$$\frac{dz_b}{dr} = \frac{-(G/c^2)\left(2z_b + \sqrt{2z_b}\right)\left[M(r) + 4\pi r^3 (m_b/v) z_b\right]}{r\left(r - \frac{2GM(r)}{c^2}\right)} \tag{6a}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 (m_b/v)\left[z_b + \sqrt{2z_b}\right] . \tag{6b}$$

The equations are solved by specifying a central mass density (from which Eq. (4a) gives the central pressure) and integrating outward over radius r until $r = R$ at which pressure $z(R) = 0$. Then R is the stellar radius and $M \equiv M(R)$ is the stellar gravitational mass as inferred by a distant observer. Besides relation (5), stability requires that M be less than the composite mass M_b of the unbound particles at rest at infinity, given by

$$M_b = \int_0^R 4\pi r^2 g_{rr}(r) m_b n_b dr = \int_0^R 4\pi r^2 (m_b/v)\sqrt{2z_b}\left(1 - \frac{2GM(r)}{rc^2}\right)^{-\frac{1}{2}} dr \tag{6c}$$

where $g_{rr}(r) = \left(1 - \frac{2GM(r)}{rc^2}\right)^{-\frac{1}{2}}$ is the radial element of the (diagonal) Schwarzschild metric tensor.

The EOS of the condensate is determined by two length parameters: λ_b (~inverse of the boson mass) and a (~strength of the boson S-wave interaction). For each pair of EOS parameters a sequence of equilibrium stellar masses is generated from the TOV equations by varying the central mass density. Figure 1 shows an example of such a sequence for parameters corresponding to neutron pairing with mass parameter $Q \equiv m_b/2m_n = 1.2$ and $a/\lambda_b = 10^6$. Figure 2 shows the radial variation in mass density and pressure for the maximum-mass star in this sequence. Although I have examined numerous cases of EOS parameters and resulting stellar sequences, I draw particular attention in this paper to solutions with $a/\lambda_b >> 1$. As illustrated in Table 1, these are solutions for which the mean particle separation within the stellar bulk is much smaller than the scattering length, yet much larger than the healing length. The first condition corresponds to the unitarity limit for which fermion behavior is universal, i.e. largely independent of the scattering potential and therefore applicable to dense nuclear matter as well as to dilute atomic gases.[3] The second condition justifies neglect of the zero-point quantum pressure (except in the proximity of the surface).

Foremost to be noted is that the densities of stable equilibrium states (left of the peak in Figure 1) fall within the range between white dwarf and neutron stars, and that equilibrium masses can far exceed the Oppenheimer-Volkoff limit.

TABLE 1: Neutron Condensate Solutions to the TOV Equations (Q = 1.2)

a/λ	M_{max}/M_{Sun}	$\mathrm{Log}(\rho)$*	R (km)	a/d_b	ξ_0/d_b
200	1.7	15.00	15	17.3	0.048
2000	5.5	14.00	47	104.5	0.020
4000	8.3	14.00	57	116.4	0.018
8000	11.7	13.70	81	184.7	0.015
12000	14.1	13.70	90	266.3	0.012
24000	19.6	13.00	158	338.4	0.011
1000000	130.4	11.70	850	4870	0.003

* ρ in kg/m^3

The stellar model presented here raises a number of fundamental issues. At present, it is not known whether neutrons will condense under sufficiently high pressures to form a BEC.[4] However, since there is strong evidence that neutron pair correlations give rise to superfluidity in the outer layers of a neutron star[5], the formation of a condensate within the denser interior is highly conceivable. This inference is supported by recent experiments on cold fermionic gases demonstrating the parametric variation between superfluid and BEC states in the unitarity limit.[6]

A second issue is whether the neutron condensate, presuming it forms under high compression, will be the denser and therefore more stable phase. A comprehensive analysis of this problem would exceed the space limits of this paper. The formal structure of the calculation is outlined in the Appendix. In brief, I have solved the TOV equations for a two-phase system, employing the Gross-Pitaevski EOS for the condensate phase and a modified Bethe-Johnson[7] EOS for the neutron phase, and have found sequences of stable states with a dense condensate core for reasonable choices of parameters.

A third and particularly noteworthy issue is the potential implication of the present work for theories of dark matter, a seemingly distinct problem that also lies at the forefront of contemporary astrophysics. The solutions to the TOV equations for a self-gravitating condensate, as reflected in Table 1, reveal a progression of increasing mass limits as $a/\lambda\!\!\!^{-}_b$ increases. The wider significance of this trend may be seen as follows. The intrinsic material property of the self-gravitating condensate enters the TOV equations (6a-c) only through the ratio m_b/v which can be re-expressed in the form $\frac{m_b}{v}=\frac{(c/\hbar)^2}{4\pi^2}\frac{m_b^3}{a}$. Thus, letting $a\rightarrow\infty$ for fixed boson mass m_b, is equivalent to letting $m_b\rightarrow 0$ for a fixed interaction a. Therefore, apart from application to cold dense stars, solutions in the unitarity limit can also be interpreted as pertaining to self-gravitating condensates of very low-mass bosons. Table 1 suggests that stable states may be generated with high gravitational masses, low mass densities, and radii on a galactic scale by selection of sufficiently low values of the boson mass. Such states could provide an apt description of galactic dark matter.

Prevailing theories attribute dark matter to various species of weakly interacting bosons with masses many times greater than a proton mass. However, although dark matter has been detected indirectly through gravitational lensing[8], experimental searches have failed to find any reproducible direct evidence of the hypothesized massive particles.[9] I have previously proposed a theory of dark matter as a condensate of very low mass bosons with a density function determined from the GP equation.[10] The predicted mass distribution, which agreed reasonably well with observations based on rotation curves of spiral galaxies, is quite different from the density profiles generated by TOV solutions such as illustrated in Figure 2. However, the low density of galactic dark matter is precisely the condition ($n_b < (8\pi a)^{-3}$) for which the quantum zero-point energy $x_Q \equiv \frac{1}{8}\lambda_b^2 \frac{(\nabla \eta_b)^2}{\eta_b}$ in Eq. (1a) and corresponding quantum pressure $z_Q = \eta_b \frac{dx_Q}{d\eta_b} - x_Q$, hitherto neglected, are expected to play a dominant role. It remains to be seen, therefore, whether a general relativistic analysis of a quantum condensate with quantum zero-point stress-energy tensor elements can provide a satisfactory description of galactic dark matter. An endeavor to answer that question is under way.

APPENDIX

Self-Gravitating Neutron and Neutron-Condensate Phases in Equilibrium

The neutron phase can be characterized by a modified Bethe-Johnson EOS

Pressure-Energy

$$x_n \equiv \frac{\varepsilon_n}{m_n c^2 n_0} = \left(\frac{z_n}{\nu - 1}\right) + \left(\frac{z_n}{\nu - 1}\right)^{1/\nu} \tag{A1}$$

Pressure-Chemical Potential $$z_n \equiv \frac{P_n}{m_n c^2 n_0} = \left(\frac{\zeta_n - 1}{\nu}\right)^{\nu/(\nu-1)} \qquad \text{(A2)}$$

expressed in terms of dimensionless quantities for energy density ε_n, pressure P_n, and chemical potential $\zeta_n \equiv \frac{\mu_n}{(m_n c^2)} = \frac{dx_n}{d\eta_n}$, where $\eta_n \equiv (n_n / n_0)$ is the dimensionless particle density, and the exponent ν and number density n_0 are characteristic parameters of the model. Typical values, based on comparison with nuclear data, are $\nu = 2.5$ and $n_0 = 10 n_{\mathrm{N}}$, where the saturation nuclear density is $n_{\mathrm{N}} = 0.15\ \mathrm{fm}^{-3}$.

The Bethe-Johnson EOS is not causal over the entire range of chemical potentials. From the relation

$$1 \geq \frac{dP_n}{d\varepsilon_n} = \frac{dz_n}{dx_n} = (\nu - 1)\left(1 - \zeta_n^{-1}\right) \geq 0 \qquad \text{(A3a)}$$

one deduces the restriction

$$\frac{\nu - 1}{\nu - 2} \geq \zeta_n \geq 1 \qquad \text{(A3b)}$$

or $3 \geq \zeta_n \geq 1$ for the parameters previously specified. Actually, the lower limit may be larger than 1 as a consequence of meeting the condition of chemical equilibrium.

The reversible transformation of unassociated neutrons into a neutron condensate, represented by the reaction $sn \leftrightarrow b$ where $s = 2$ for pairing, is subject to baryon conservation from which the condition of chemical equilibrium follows as an equality of chemical potentials

$$s\mu_n = \mu_b \quad \text{or} \quad \zeta_b = \zeta_n / Q \qquad \left(Q \equiv m_b / s m_n\right). \qquad \text{(A4)}$$

The transition pressure P_c at which the condensate phase begins to form is determined from Eqs. (4b) and (A2) by the equality $P_b = P_n$,

$$(\nu - 1)\left(\frac{\zeta_n - 1}{\nu}\right)^{\nu/(\nu-1)} = \left(\frac{sQ}{2 n_0 \nu}\right)\left(\frac{\zeta_n}{Q} - 1\right)^2. \qquad \text{(A5)}$$

For the condensate to be the stable phase in the core, it must be the denser phase at pressures $P > P_c$, as a consequence of which it may be shown that $Q > 1$, and Q becomes the lower limit in Eq. (A3). If Q is too large, $\zeta_n(P_c)$ exceeds the upper limit of the causal region of the Bethe-Johnson EOS (a problem presumably circumvented by a more comprehensive theory of compressed nuclear matter). Choosing for illustrative purposes $Q = 1.2$ and $a/\lambda\!\!\!^{-}_b = 200$ yields $\zeta_n(P_c) = 2.866$ and $P_c = 2.08 \times 10^{35}$ Pa. As points of comparison, note that the pressure (according to Newtonian gravity) at the center of a solar-mass neutron star of radius 10 km is $\frac{3}{8\pi}\frac{GM^2}{R^4} = 3.2 \times 10^{33}$ Pa, and that a neutron gas of density 1 particle per cubic Compton wavelength yields a characteristic pressure $\frac{mc^2}{\lambda_n^3} = \frac{m^4c^5}{h^3} = 6.5 \times 10^{34}$ Pa. The mass, size, and density distribution of the two-phase star is obtained by integrating the TOV equation with the neutron-phase EOS for $P < P_c$ and the condensate-phase EOS for $P > P_c$.

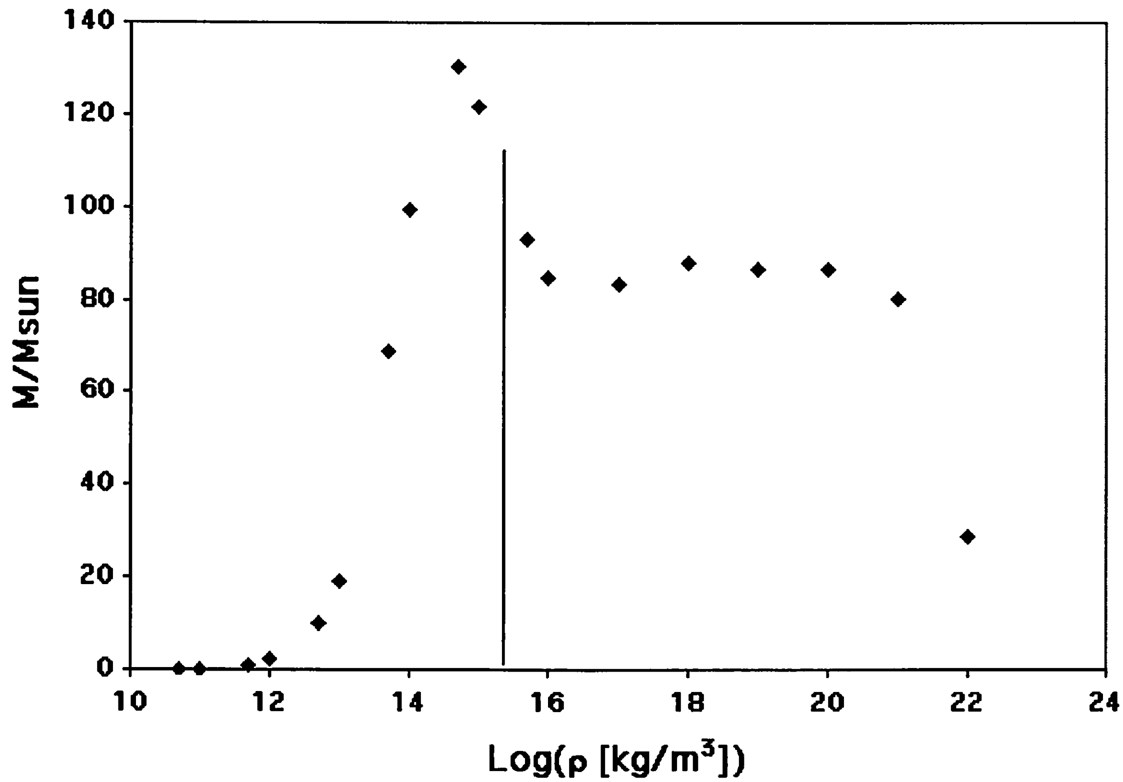

Figure 1: Equilibrium stellar masses (in units of solar mass) as a function of the logarithm of the central density for $a/\lambda_b = 10^6$ and $Q = 1.2$. Stars to the right of the vertical line are in unstable equilibrium because $M > M_b$.

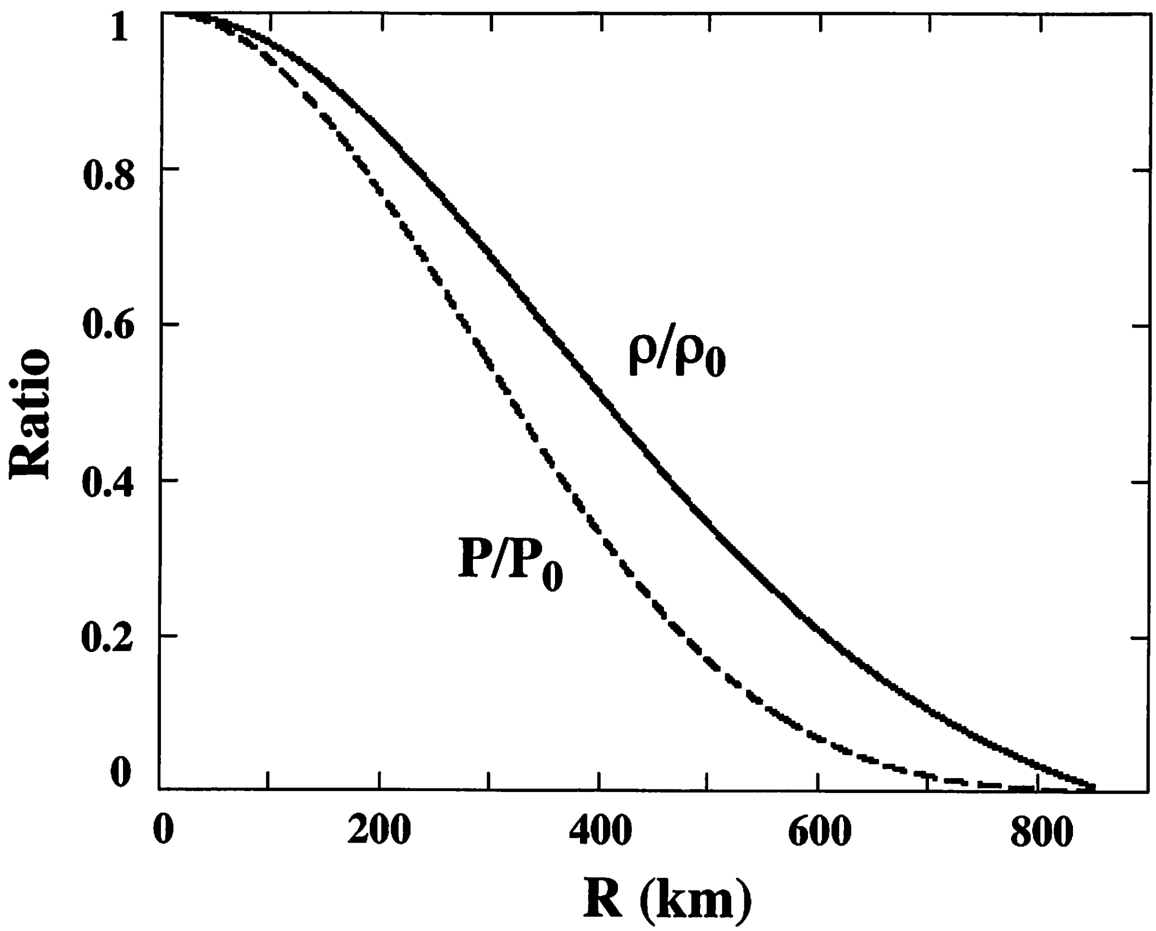

Figure 2: Variation in mass density (relative to the central density $\rho_0 = 5 \times 10^{14}$ kg/m^3) and pressure (relative to the central pressure $P_0 = 5.1 \times 10^{30}$ Pa) for condensate stars with $a/\lambda\!\!\!^{-}_b = 10^6$ and $Q = 1.2$.

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