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Fermion Condensation In A Relativistic Degenerate Star: Arrested Collapse and Macroscopic Equilibrium*

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Summary

Fermionic Cooper pairing leading to BCS-type hadronic superfluidity is believed to account for periodic variations ("glitches") and subsequent slow relaxation in spin rates of neutron stars. Under appropriate conditions, however, fermions can also form a Bose-Einstein condensate of composite bosons. Both types of behavior have recently been observed in tabletop experiments with ultra-cold fermionic atomic gases. Since the behavior is universal (i.e. independent of atomic potential) when the modulus of the scattering length greatly exceeds the separation between particles, one can expect analogous processes to occur within the supradense matter of neutron stars. In this paper I show how neutron condensation to a Bose-Einstein condensate, in conjunction with relativistically exact expressions for fermion energy and degeneracy pressure and the relations for thermodynamic equilibrium in a spherically symmetric spacetime with Schwarzschild metric, leads to stable macroscopic equilibrium states of stars of finite density, irrespective of mass.

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The collapse of a massive relativistic degenerate star to a stellar black hole poses what is perhaps the most serious challenge to the laws of physics as they are currently understood. Prevailing theory, starting with the first comprehensive analysis of the problem by Oppenheimer and his group [1], holds that all matter and energy, once having passed through the event horizon, fall irretrievably into a central singularity. To the extent that a black hole is a real physical object and not merely a mathematical solution to the differential equations of general relativity, it is difficult to believe that 10^{30} or more kilograms of matter can actually collapse to a geometric point (or, according to some theories, to a region of Planck length size $\sim 10^{-35}$ m). Clearly something important has been omitted from the relevant physics.

During the past few years there have been several attempts to stabilize the collapse of a star beyond the Oppenheimer-Volkoff (O-V) mass limit by taking account of long-range magnetic interactions among nucleons [2], by recognition of a vacuum quantum field process (particle resorption) [3] complementary to the process responsible for Hawking radiation [4], and by appeal to loop quantum gravity to suppress the formation of an event horizon [5].

Very recent advances in the investigation of degenerate fermi gases at ultra-low temperature [6][7][8][9], however, suggest a new possibility by means of which known physical laws (in contrast to as yet hypothetical quantizations of gravity) may prevent the singular collapse of a degenerate star. Currently under intensive theoretical and experimental investigation, ultra-cold gases of fermionic atoms (e.g. ^{40}K and ^{6}Li) with magnetically tunable interactions provide a means of exploring the predicted transition between Bardeen-Cooper-Schrieffer (BCS) superfluidity and Bose-Einstein condensation

(BEC) into composite bosons. In such systems, the modulus of the scattering length can be made much larger than the mean atomic separation, a condition believed to hold in neutron stars. Thus, degenerate atomic fermi gases provide experimentally accessible model systems for exploring processes that can occur in dense nuclear matter [10][11][12].

The existence of neutron and proton BCS-type superfluids in neutron stars has long been supposed to account for sudden period changes ("glitches") and their subsequent slow relaxation [13], although questions remain concerning the exact nature and distribution of hadronic superfluids [14]. Fermionic BCS and BEC behavior represent two extremes in the condensation of fermions, depending on whether the scattering length asymptotically approaches negative or positive infinity [15], whereupon the behavior becomes universal, i.e. independent of the details of the atomic potential. In the first case, which entails formation of Cooper pairs, the underlying Fermi statistics play an essential role[16]; in the second case, all fermions are bound into quasi-molecular composite bosons with no fermionic degree of freedom remaining. Both kinds of fermionic behavior have been observed in the recent experiments on cold fermionic gases. Since there is compelling evidence that hadronic BCS superfluids exist in neutron stars, it is not unreasonable to consider that fermionic matter may also condense to a BEC under the conditions of supranuclear density and strong magnetic fields encountered within degenerate stars over the O-V limit.

In this paper I show that nucleon (or, if hadrons disrupt, then quark) condensation to a BEC, in conjunction with relativistically exact expressions for fermion energy and degeneracy pressure and the relations for thermodynamic equilibrium in a spherically symmetric spacetime with Schwarzschild metric, lead to stable macroscopic equilibrium states of degenerate stars irrespective of mass.

To understand qualitatively how fermion condensation may stabilize stellar collapse, one must recall why standard physical arguments lead to no equilibrium in the first place. As the fermionic matter in a collapsing degenerate star becomes relativistic, the dependence of the degeneracy pressure on the density changes from a 5/3 power dependence to a 4/3 power dependence. In consequence of this softening, the hydrostatic balance between degeneracy pressure and weight is broken, and, in the absence of any other mechanism to stiffen the pressure law, the matter continues to collapse to a singularity. In the process presented here, however, the fermionic fluid of neutrons (or quarks), compressed to densities well in excess of normal nucleon density (i.e. neutron mass divided by the cube of the neutron Compton wavelength), undergoes a reversible transformation to a BEC of composite bosons, the equilibrium concentrations of which depend on the density, and therefore size, of the collapsing star. As neutrons are removed from the fermion phase, the Fermi energy consequently falls, and the degeneracy pressure eventually stiffens. I will show that condensation to a BEC leads again to hydrostatic equilibrium with an equilibrium radius comparable to that of a neutron star or quark star.

As the simplest model to illustrate the preceding principles, I consider a uniformly dense two-component system of degenerate neutrons of mass m_n and Bose-Einstein condensate of composite bosons of mass m_b . The internal energy function of this system can be written as [17][18]

$$U = \left[m_n c^2 g(y_F) V / 8\pi^2 \lambda_n^2 \right] + N_b m_b c^2 + \left(1 - f(x)\right) M c^2 = M c^2 \qquad (1)$$

where the first bracketed term is the fermion energy U_f of N_n degenerate neutrons in volume V,

$$y_F = p_F / m_n c = \left(3\pi^2 N_n / V\right)^{1/3} \lambda_n \quad (2a)$$

is the ratio of the fermi momentum to the product of neutron mass and light speed, $\lambda_n = \hbar / m_n c \sim 0.21$ fm is the reduced Compton wavelength of the neutron, and the function *g(y)* is defined by

$$g(y) = y(2y^2 + 1)\sqrt{y^2 + 1} - \sinh^{-1}(y). \quad (2b)$$

The second term is the ground-state energy U_b of the Bose-Einstein condensate of N_b bosons of mass m_b. The third term is the gravitation energy U_g of the entire system of mass *M* defined by

$$M \equiv \int_0^R 4\pi r^2 \rho dr = 4\pi R^3 \rho / 3 \quad (3a)$$

and evaluated for uniform density ρ. The function *f(x)*, which defines a spherical volume of radius *R* in the Schwarzschild spacetime as

$$V = (4\pi R^3 / 3) f(x) , \quad (3b)$$

takes the form

$$f(x) = \left(3 / 2x^2\right)\left[\frac{\sin^{-1} x}{x} - \sqrt{1 - x^2}\right], \quad (3b)$$

where the square of the argument is the ratio of the Schwarzschild radius to the stellar radius

$$x^2 = 2GM / Rc^2 = R_S / R. \quad (3c)$$

From Eq. (1) there follows the relation between mass density ρ and number densities of the fermions ($n_n = N_n / V$) and bosons ($n_b = N_b / V$):

$$\rho = \frac{m_n g(y_F)}{8\pi^2 ƛ_n^3} + n_b m_b. \qquad (4)$$

Differentiation of U with respect to volume and particle numbers, under the assumption of total mass conservation, leads to the thermodynamic first-law relation

$$dU = -(P_f + P_g)dV + (\mu_n + m_n\phi)dN_n + (\mu_b + m_b\phi)dN_b \qquad (5a)$$

in which the fermion degeneracy pressure and gravitational pressure are respectively

$$P_f = -\left(\partial U_f / \partial V\right)_N = \left(m_n c^2 / 24\pi^2 ƛ_n^3\right)\left[y_F\left(2y_F{}^2 - 3\right)\sqrt{y_F{}^2 + 1} + 3\sinh^{-1} y_F\right] \qquad (5b)$$

$$P_g = -\left(\partial U_g / \partial V\right)_N = \rho c^2 \frac{x f'(x)}{x f'(x) - 6f(x)} \qquad (f' = df/dx), \qquad (5c)$$

the fermion and boson chemical potentials are respectively

$$\mu_n = \left(\partial U_f / \partial N_n\right)_V = m_n c^2 \sqrt{1 + y_F^2} \qquad (5d)$$

$$\mu_b = \left(\partial U_b / \partial N_b\right)_V = m_b c^2 , \qquad (5e)$$

and the gravitational potential is defined by

$$\phi(x)/c^2 = \left(\partial U_g / \partial Mc^2\right)_N = -\left[\frac{1}{\sqrt{1 - x^2}} - 1\right]. \qquad (5f)$$

In deriving Eq. (5a) and the succeeding expressions, I have neglected the entropy contribution (TdS) since the mean temperature T of the system, although it can be numerically high (e.g. $T \sim 10^7$ K) initially, is negligibly small compared to the Fermi temperature $T_F = \mu_n / k_B$ where k_B is Boltzmann's constant ($T_F > 10^{13}$ K for a solar-mass neutron star) and the BEC-BCS cross-over temperature ($T_c \sim 0.15\, T_F$ [15]). For all practical purposes, therefore, the degenerate matter, both fermionic and bosonic, is effectively at a temperature of absolute zero.

Consider next the reversible transformation $sn \leftrightarrow b$, whereby an even number s of neutrons correlate to produce a composite boson with mass parameter $Q = m_b / sm_n$. Studies of neutron superfluidity in neutron stars [19] have generally assumed $s = 2$, but other correlations are conceivable given the greater densities occurring in stars collapsing to black holes. The preceding transformation law leads to the differential change in particle number $dN_b = -dN_n / s$, which, upon substitution into Eq. (5a), results in the equality

$$s(\mu_n + m_n\phi) = \mu_b + m_b\phi \tag{6}$$

at equilibrium ($dU = 0$, $d^2U < 0$). Eq. (6) is a generalization of the equality of chemical potentials for systems in a weak gravitational potential [20]. Likewise, hydrostatic equilibrium requires

$$P_f + P_g = 0. \tag{7}$$

Substitution of Eqs. (5d,5e,5f) into Eq. (6) leads to the equilibrium relation between the neutron Fermi parameter y_F and the Schwarzschild parameter x

$$y_F = \left[\left[Q + (Q-1)\left(1 - \frac{1}{\sqrt{1-x^2}}\right)\right]^2 - 1\right]^{1/2} \qquad (0 \le x^2 \le 3/4), \tag{8a}$$

or equivalently

$$x^2 = R_S / R = 1 - \left(\frac{Q-1}{2Q - 1 - \sqrt{1+y_F^2}}\right)^2 \qquad \left(0 \le y_F \le \sqrt{Q^2 - 1}\right) \tag{8b}$$

from which the variation in neutron density as a function of stellar radius can be calculated (Eq. 2a)

$$n_n = \left(y_F(x)/\lambdabar_n\right)^3/3\pi^2 \ . \tag{8c}$$

The density of the composite bosons then follows from the relation

$$n_b = \frac{1}{s}\left(\frac{V_0}{V}n_n^0 - n_n\right), \tag{8d}$$

where n_n^0 is the initial fermion density at the volume V_0 just before onset of the fermion-BEC transition. The requirement that y_F and x^2 be real-valued and positive sets the limits of variation shown in Eqs. (8a,b)

The equilibrium value of x_{eq} to be used in Eq. (8c) is determined by solution to the pressure-balance equation (7). In this regard, it is necessary to take note of the uniquely quantum mechanical implications of having a self-gravitating BEC. As is well known, a boson condensate at effectively zero temperature exerts no pressure because all particles are in a state of zero relative momentum [21]. (The pressure of a nonrelativistic Bose gas approaches zero with temperature as $T^{5/2}$ independent of volume.) What must also be realized, however, is that a self-gravitating condensate in its ground state exerts *no weight.* Were the particles of the condensate to fall inwards towards the center of the star, they could not maintain a ground state of zero relative momentum, yet the effective temperature of the boson condensate is so far below the BEC critical temperature that the condensate cannot be excited [22]. The outcome, therefore, is that the condensate, in accordance with the quantum uncertainty principle, cannot collapse to a singularity, a characteristic already noted in a model of galactic dark matter as a BEC of low-mass bosons [23]. Only the fermionic matter (the neutron or quark fluid) within the star continues infalling, until hydrostatic equilibrium is attained. In view of the fact that the self-gravitating BEC exerts no weight, it is not the total density ρ that must appear in Eq. (5c) for the gravitational pressure, but only the fermion contribution to the density. Thus, in place of Eqs. (4) and (5c), one must instead employ in Eq. (7) the pressure

$$P_g = \rho_n c^2 \frac{x f'(x)}{x f'(x) - 6f(x)} \tag{9a}$$

with density

$$\rho_n = m_n g(y_F)/8\pi^2 \lambda_n^3 . \tag{9b}$$

Examination of Eq. (8a) shows that y_F indeed decreases (i.e. neutron density falls) as x increases (i.e. the star collapses) over the allowed range of variation. Eq. (7), which is highly nonlinear in the variable x, can be solved numerically by computer for initially specified parameters: e.g. maximum Fermi parameter y_F^o, condensation factor s, and condensate mass parameter Q. Figures 1, 2, and 3 illustrate the principal features of such a solution for a relativistic degenerate star for which fermion condensation occurs with condensate parameters $s = 2$, and $Q = 4$. As the stellar radius decreases, the equilibrium neutron density falls and the equilibrium composite-boson density rises (Figure 1A), although the number of composite particles can never exceed the initial number of neutrons (Figure 1B). Both degeneracy pressure and gravitational pressure fall in magnitude, achieving hydrostatic equilibrium at a radius of approximately 1.5 R_S (Figure 2). At equilibrium, the neutron Fermi parameter y_F is approximately 1.5.

Several points are worth noting explicitly. First, for relativistic degenerate stars sufficiently dense that neutrons condense to form a BEC, the terminal equilibrium state is not a black hole with central singularity, but a new stellar species of size marginally larger than the theoretical Schwarzschild radius. This is the case irrespective of the condensate parameters, the values of which a more complete theory of supradense nuclear matter would presumably specify. Second, the case $Q = 0$ effectively corresponds to the previously reported process of gravitationally-induced particle resorption into the vacuum [3]. Third, although the preceding analysis was implemented for intact neutrons, the basic principles apply as well to the condensation of quarks in neutron stars with hadron disruption. In such a case, however, the expressions determining equilibrium composition must be generalized to take account of different quark flavors and the quantum chromodynamical modification of the quark chemical potential [24]. Fourth, a more complete analysis than can be presented here must also take account of the effect on BEC formation of the long-range magnetic interactions among neutrons (or quarks). Previous investigations of magnetic interactions in degenerate stars collapsing under Newtonian gravity [2] have led to macroscopic stable states marginally smaller in size than the Schwarzschild radius. Such a star would qualify as a black hole although with no singularity. Further work is now under way to see whether magnetic interactions in conjunction with fermionic BEC formation in a Schwarzschild spacetime lead to similar results.

The question arises of how one can determine observationally whether a degenerate star of mass greater than the O-V limit is a black hole (with an event horizon and matter and energy concentrated at the interior singularity) or a star, such as predicted here, with neutrons (or quarks) in equilibirum with a BEC of composite bosons. The usual distinguishing criteria of mass (estimated from orbital data for stars in binary associations or from gravitational microlensing events) and size (estimated from fluctuations in X-ray emissions by accreting matter), may be inconclusive. More likely to succeed would be astronomical observations that can reveal the physical characteristics of the stellar surface, in particular to determine its emissivity and presence of a a physical crust.

In concluding this paper, it is perhaps pertinent to recall the desperate comment made by Sir Arthur Eddington with respect to Chandrasekhar's discovery of his eponymous limit on white dwarf masses and the irreversible collapse that it implied [25]: "Various accidents may intervene to save the star, but I want more protection than that. I think there should be a law of nature to prevent a star from behaving in this absurd way." If it can be convincingly demonstrated that self-gravitating supradense degenerate hadronic matter *must* indeed undergo a transition to a BEC state, then the mechanism herein proposed will at long last provide that "law of nature" that Eddington called for. It is fascinating to contemplate how a rare form of matter that can be made only with great difficulty and exists in relatively few terrestrial laboratories may turn out to be one of the most common components of the cosmos and central to the gravitational stability of degenerate stars.

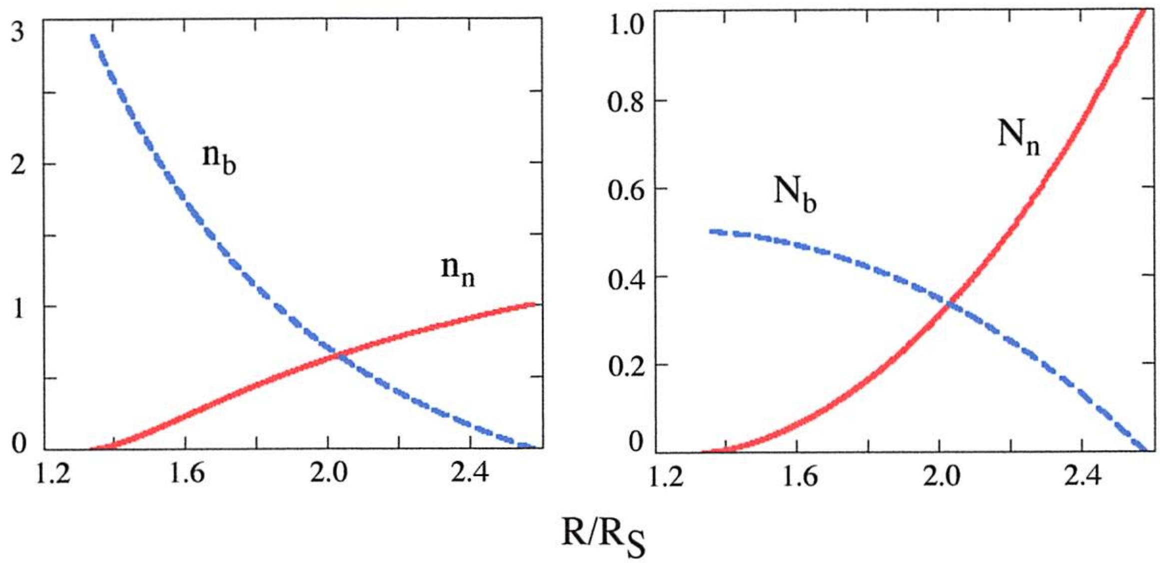

Figure 1: (A) Variation in neutron and composite boson densities with radius (in units of Schwarzschild radius R_S). The densities are in units of the initial neutron density when BEC formation begins.. (B) Variation in neutron and composite-boson particle numbers (in units of initial number of neutrons). Condensate parameters are $s = 2$ and $Q = 4$.

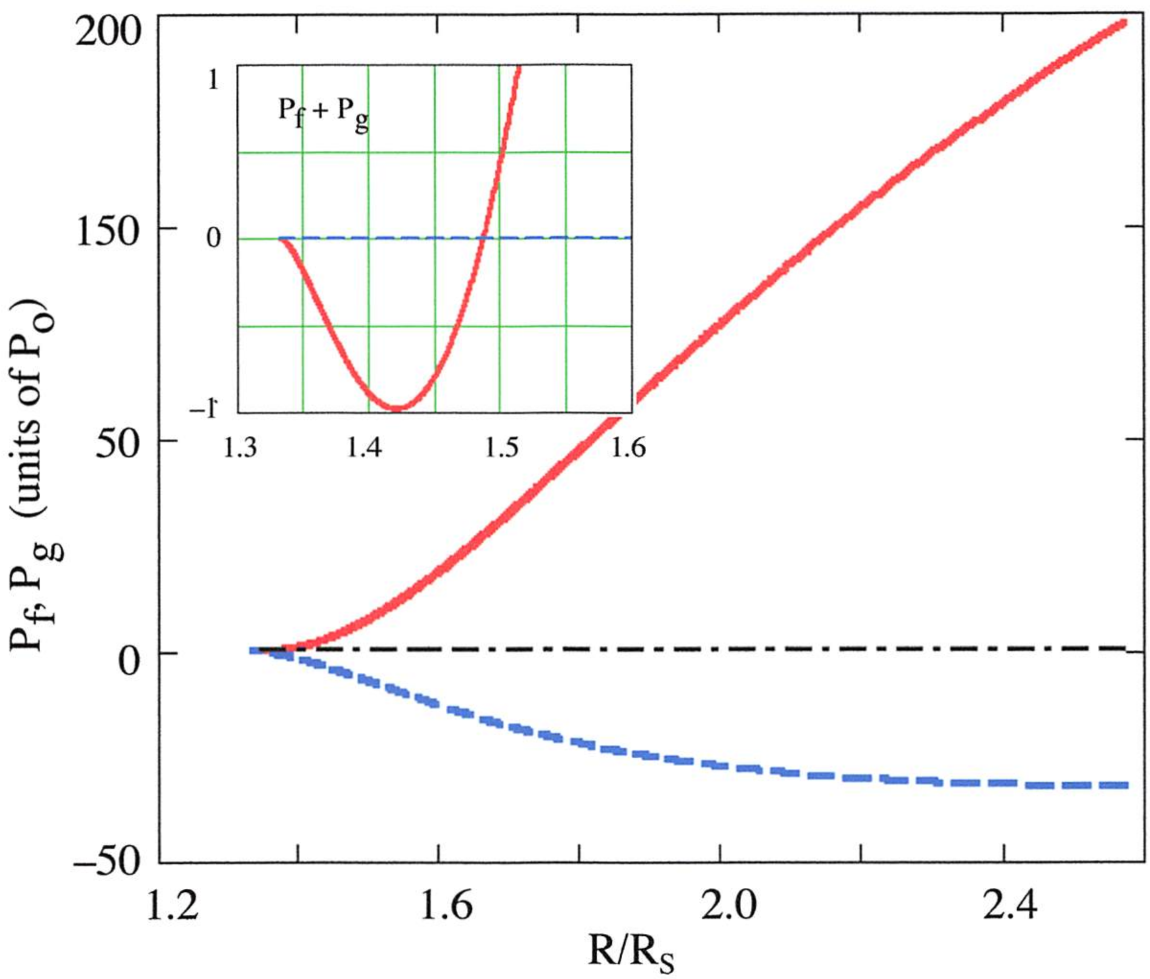

Figure 2: Variation in fermion degeneracy pressure and gravitational pressure with radius (in units of Schwarzschild radius) for condensate parameters s=2, Q=4. The insert (total pressure vs radius) shows the region where hydrostatic equilibrium is attained at R ~ 1.5 R_S. Pressures are expressed in units of $P_0 = m_n c^2 / 24\pi^2 \lambda\!\!\!^{-}{}_n^3 = 6.9 \times 10^{34}$ Pa.

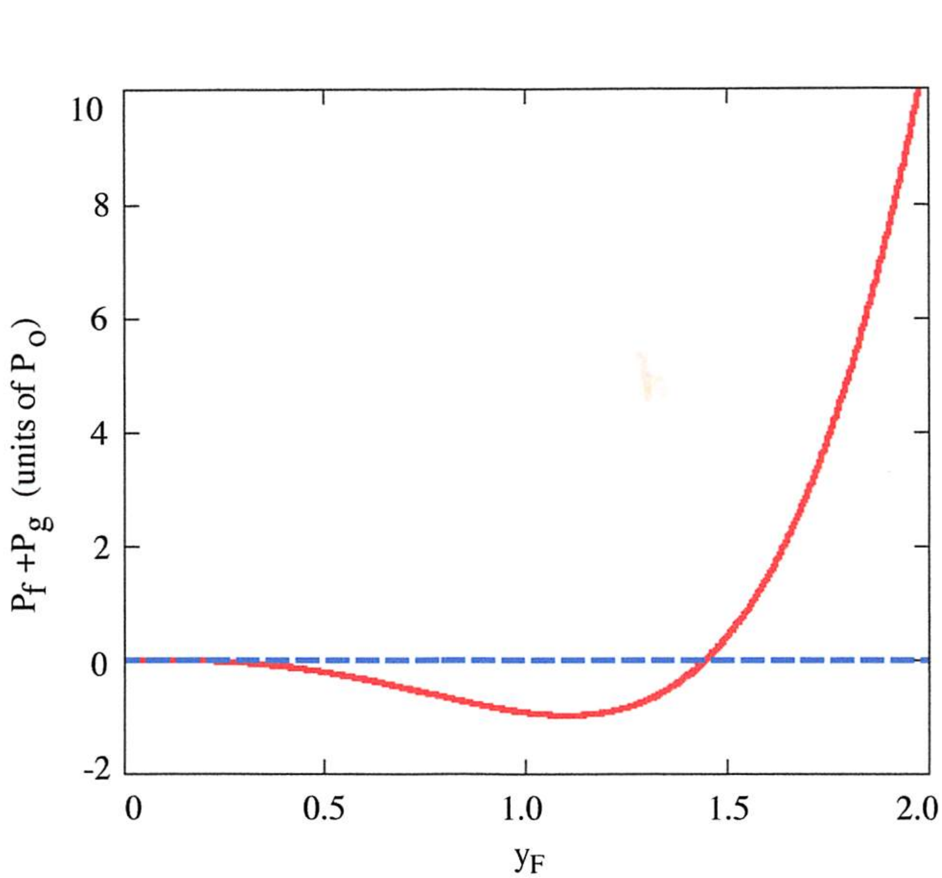

Figure 3: Variation in total (degeneracy + gravitational) pressure with Fermi momentum parameter y_F for condensate parameters s=2, Q=4. Pressures are expressed in units of $P_0 = m_n c^2 / 24\pi^2 \lambda_n^3 = 6.9 \times 10^{34}$ Pa. Hydrostatic equilibrium is attained at approximately $y_F \sim 1.5$.

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