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The Lamplighter Group

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THE LAMPLIGHTER GROUP

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Foreword

This thesis is a survey of The Lamplighter Group following a set of notes as written by Professor Jennifer Taback of Bowdoin College. Topics in the preliminary section were discussed and reinforced in the course material of *Special Topics in Group Theory* as overseen by Professors Melanie Stein and Keith Jones.

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Contents

Chapter 1

Preliminaries

1.1 Free Groups

The goal of this section is to provide the reader with a thorough understanding of free groups, a topic whose comprehension is imperative for the discussion of Group Presentation and, ultimately, the discussion of the Presentation of the Lamplighter Group.

We begin with a series of definitions.

Definition 1.1.1. Let S be a set of symbols. Then a *word* w in S is a finite sequence $s_1 \cdots s_k$ where $\forall i,\ s_i \in S$. We say the length of such a word, denoted $l(w)$ is k. Let $w_\emptyset$ denote the empty word, the word of length 0.

Consider the set $T = \{b_1, b_2, b_3, b_4\}$ Then $b_1b_2b_3b_4$, $b_4b_3b_2b_1$, b_1b_3, b_2b_4, b_1, and $w_\emptyset$ are all words in T.

Definition 1.1.2. Given $w = s_1 \cdots s_n$, a word in S, we say $w' = s_i \cdots s_j$ is a sub-word of w for all $1 \leq i < j \leq n$. Furthermore, w' is a *prefix* of w if $i = 1$ and w' is a *suffix* of w if $j = n$.

Definition 1.1.3. For S, a set of symbols, we define $S^{-1} = \{s^{-1} | s \in S\}$. Furthermore, $\forall a \in S \cup S^{-1}$, we define

$$a^{-1} = \begin{cases} t^{-1} & \text{if } \exists t \in S \text{ with } a = t \\ t & \text{if } \exists t \in S^{-1} \text{ with } a = t \end{cases}$$

Note that this definition does not refer to formal inverses. Rather, it refers simply to the set formed by superscripting symbols in S with -1.

Definition 1.1.4. For S, a set of symbols, we define $(S \cup S^{-1})^*$ to be the set of all words in $S \cup S^{-1}$.

Again, we consider the set $T = \{b_1, b_2, b_3, b_4\}$. Then the following are words are contained in $(T \cup T^{-1})^*$:

$$\begin{array}{ccc} b_1 \cdots b_n & b_1^{-1}b_2 & b_1b_1b_1b_1 \\ b_1^{-1} \cdots b_4^{-1} & b_2 & b_1^{-1}b_1^{-1}b_1^{-1}b_1^{-1} \\ b_1b_2^{-1}b_3b_4^{-1} & w_\emptyset & b_1b_1^{-1}b_1b_1^{-1}b_1b_1^{-1} \end{array}$$

We define operation amongst words in the definition below.

Definition 1.1.5. Let $w = s_1 \cdots s_k$, $v = t_1 \cdots t_j$ be words in S. Then we define wu to be the concatenated word $wu = s_1 \cdots s_k t_1 \cdots t_j$. Note that $l(wu) = l(w) + l(u)$.

Definition 1.1.6. Let S be a set of symbols and let $w, v \in (S \cup S^{-1})^*$. Then we define the following relations:

- $wss^{-1}v \to wv$ and $ws^{-1}sv \to wv$ for any $s \in S$.
- $wv \leftarrow wss^{-1}v$ and $wv \leftarrow ws^{-1}sv$ for any $s \in S$ (the inverse of $\to$).
  - Note that $w \leftarrow v$ implies $v \to w$.
- $w \leftrightarrow v$ if and only if either $w \leftarrow v$ or $w \to v$ (the symmetric closure about $\leftarrow$).
  - Note that $w \leftrightarrow v$ implies $v \leftrightarrow w$.
- $w \xrightarrow{*} v$ if and only if either $w = v$ or $\exists\ w_1, \ldots, w_k \in (S \cup S^{-1})^*$ with $w = w_1 \to \cdots \to w_k = v$ (the reflexive, transitive closure about $\to$).
- $w \overset{*}{\leftrightarrow} v$ if and only if either $w = v$ or $\exists w_1, \ldots, w_k \in (S \cup S^{-1})^*$ with with $w = w_1 \leftrightarrow \cdots \leftrightarrow w_k = v$ (the reflexive, symmetric, and transitive closure about $\to$).

Claim 1.1.7. $\overset{*}{\leftrightarrow}$ is an equivalence relation.

Proof. To show $\overset{*}{\leftrightarrow}$ is an equivalence relation, we must show reflexivity, symmetry, and transitivity.

Note that $w \overset{*}{\leftrightarrow} w$ by the definition of $\overset{*}{\leftrightarrow}$, giving reflexivity.

To show symmetry, let $w, v \in (S \cup S^{-1})^*$ such that $w \overset{*}{\leftrightarrow} v$. We wish to show $v \overset{*}{\leftrightarrow} w$. As $w \overset{*}{\leftrightarrow} v$, there exists a finite sequence $w_1, \ldots, w_k \in (S \cup S^{-1})^*$ such that $w = w_1 \leftrightarrow \cdots \leftrightarrow w_k = v$. However, by the definition of $\leftrightarrow$, $\forall i$ with $1 \leq i \leq k-1$, $w_i \leftrightarrow w_{i+1}$ implies $w_{i+1} \leftrightarrow w_i$. Thus, $v = w_k \leftrightarrow \cdots \leftrightarrow w_1 = w$, giving $v \overset{*}{\leftrightarrow} w$ as desired.

To show transitivity, let $w, v, u \in (S \cup S^{-1})^*$ such that $w \overset{*}{\leftrightarrow} v$ and $v \overset{*}{\leftrightarrow} u$. We wish to show $w \overset{*}{\leftrightarrow} u$. As $w \overset{*}{\leftrightarrow} v$, there exists a sequence $w_1, \ldots, w_k \in (S \cup S^{-1})^*$ such that $w = w_1 \leftrightarrow \cdots \leftrightarrow w_k = v$. Similarly, as $v \overset{*}{\leftrightarrow} u$, there exists a finite sequence $v = v_1 \leftrightarrow \cdots \leftrightarrow v_j = u$. Thus, since $w_{k-1} \leftrightarrow w_k = v = v_1$, we may write

$$w = w_1 \leftrightarrow w_{k-1} \leftrightarrow v_1 \leftrightarrow \cdots \leftrightarrow v_j = u$$

giving $w \overset{*}{\leftrightarrow} u$ as desired. $\square$

If $w \overset{*}{\leftrightarrow} v$, then $\exists w_1, \ldots, w_k \in (S \cup S^{-1})^*$ with with $w = w_1 \leftrightarrow \cdots \leftrightarrow w_k = v$. This is to say that there exists a finite sequence of $k-1$ modfications to bring w to v. Each of these modifications is simply the insertion or deletion of an inverse pair somewhere in the word. As $\overset{*}{\leftrightarrow}$ is an equivalence relation, we may consider the equivalence classes of $\overset{*}{\leftrightarrow}$. We give notation to these equivalence classes below.

Definition 1.1.8. Let S a set of symbols and $w \in (S \cup S^{-1})^*$. Then we denote $[w]$ as the equivalence class of w under $\overset{*}{\leftrightarrow}$. That is, $[w] = \{v \in (S \cup S^{-1})^* | w \overset{*}{\leftrightarrow} v\}$.

Having completed this series of preliminary definitions, we go on next to define F_n, the free group of rank n. As with any group, we define for F_n both an underlying set and an operation. We do so respectively in the following two definitions.

Definition 1.1.9. Let $S = \{s_1, \ldots, s_n\}$ be a set. Then F_n, the free group of rank n with basis S, is the set of equivalence classes of words in $(S \cup S^{-1})^*$ under $\overset{*}{\leftrightarrow}$. In other words,

$$F_n = \{\ [w]\ | w \in (S \cup S^{-1})^*\}.$$

Definition 1.1.10. Let $[w], [v] \in F_n$. Then we define $[w][v] = [wv]$.

We must show that the operation described above is well defined. Before we do so, however, we must prove the following lemmas.

Lemma 1.1.11. Let S be a set. Then for all $w, v, x, y \in (S \cup S^{-1})^*$, $w \overset{*}{\leftrightarrow} v$ implies $xwy \overset{*}{\leftrightarrow} xvy$.

Proof. Let $w, v \in (S \cup S^{-1})^*$ such that $w \overset{*}{\leftrightarrow} v$. Then $\exists w_1, \ldots, w_k \in (S \cup S^{-1})^*$ such that $w = w_1 \leftrightarrow \cdots \leftrightarrow w_k = v$. We will employ proof by induction, inducting on the value of k.

For our base case of $k = 1$, note that $w = w_1 = v$, giving $w = v$, and as $w \leftrightarrow w$ by the definition of $\leftrightarrow$, $w \overset{*}{\leftrightarrow} w$, giving $w \overset{*}{\leftrightarrow} v$ as desired.

For our inductive case, assume that the claim is true for $k = n$ and let $w \overset{*}{\leftrightarrow} v$ such that $\exists w_1, \ldots, w_{n+1} \in (S \cup S^{-1})^*$ with $w = w_1 \leftrightarrow \cdots \leftrightarrow w_{n+1} = v$. Note that this gives us $w \overset{*}{\leftrightarrow} w_n$, and by our inductive hypothesis, this implies $xwy \overset{*}{\leftrightarrow} xw_ny$. As $w_n \leftrightarrow w_{n+1}$, w_{n+1} results from inserting or deleting an inverse pair in w_n. As concatenating w_n with x on the left and y on the right will have no effect on this insertion or deletion, we may write $xw_ny \leftrightarrow xw_{n+1}y = xvy$, giving $xw_ny \overset{*}{\leftrightarrow} xvy$. Therefore, since $\overset{*}{\leftrightarrow}$ is transitive and both $xwy \overset{*}{\leftrightarrow} xw_ny$ and $xw_ny \overset{*}{\leftrightarrow} xvy$, we have $xwy \overset{*}{\leftrightarrow} xvy$ as desired. □

Having proven this lemma, we are now able to prove the desired claim regarding operation in F_n.

Claim 1.1.12. Operation in F_n is well defined.

Proof. Let $w_1, w_2, v_1, v_2 \in (S \cup S^{-1})^*$ such that $[w_1] = [w_2]$ and $[v_1] = [v_2]$. We wish to show $[w_1v_1] = [w_2v_2]$.

Since $[w_1] = [w_2]$, we have that $w_1 \overset{*}{\leftrightarrow} w_2$. Thus, by Lemma 1.1.11, $w_1v_1 \overset{*}{\leftrightarrow} w_2v_1$. Similarly, as $[v_1] = [v_2]$, we have that $v_1 \overset{*}{\leftrightarrow} v_2$. Again, by the same lemma, this gives $w_2v_1 \overset{*}{\leftrightarrow} w_2v_2$. Thus, since $\overset{*}{\leftrightarrow}$ is an equivalence relation and is therefore transitive, since $w_1v_1 \overset{*}{\leftrightarrow} w_2v_1$ and $w_2v_1 \overset{*}{\leftrightarrow} w_2v_2$, we have $w_1v_1 \overset{*}{\leftrightarrow} w_2v_2$, which gives $[w_1v_1] = [w_2v_2]$ as desired. □

Having proven that the specified operation in F_n is well defined, we now prove that F_n is in fact a group.

Theorem 1.1.13. Let $S = \{s_1, \dots, s_n\}$ be a set. Then F_n, the free group of rank n with basis S is a group.

Proof. To show F_n is a group, we must show closure, associativity, the existence of an identity element, and the existence of inverses.

To show closure, let $[w], [v] \in F_n$. We wish to show that $[w][v] = [wv] \in F_n$. As $[w], [v] \in F_n$, $w, v \in (S \cup S^{-1})^*$. Thus, the concatenation of these words $wv \in (S \cup S^{-1})^*$ as well, giving that $[wv] \in F_n$.

To show associativity, let $[w], [v], [u] \in F_n$. We must show $([w][v])[u]) = [w]([v][u])$. Note that word concatenation amongst words in $(S \cup S^{-1})^*$ is associative. Thus, we may write

$$([w][v])[u]) = [wv][u] = [(wv)u] = [w(vu)] = [w][vu] = [w]([v][u])$$

as desired.

To show the existence of an identity element, note that $w_\emptyset \in (S \cup S^{-1})^*$ and consider $[w_\emptyset] \in F_n$. Let $[w] \in F_n$. We wish to show $[w][w_\emptyset] = [w_\emptyset][w] = [w]$. Note since $w \in (S \cup S^{-1})^*$, $ww_\emptyset = w_\emptyset w = w$. Thus, we may write

$$[w][w_\emptyset] = [ww_\emptyset] = [w] \qquad \text{and} [w_\emptyset][w] = [w_\emptyset w] = [w]$$

as desired.

To show the existence of inverses, let $[w] \in F_n$. We must show that there exists $[u] \in F_n$ such that $[w][u] = [u][w] = [w_\emptyset]$. As $[w] \in F_n$, we may write $w = a_1 \cdots a_j$ where $\forall i,\ a_i \in S \cup S^{-1}$. Consider $u = a_j^{-1} \cdots a_1^{-1}$. We will employ proof by mathematical induction, inducting on the value of j.

For our base case, consider $j = 1$. Then $w = a_1$ and $u = a_1^{-1}$, giving $wu = a_1a_1^{-1} \overset{*}{\leftrightarrow} w_\emptyset$. As $wu \overset{*}{\leftrightarrow} w_\emptyset$, $[wu] = [w][u] = [w_\emptyset]$ as desired.

For our inductive case, assume that the claim is true for $j = n$. That is, if $w' = b_1 \cdots b_n$ where $\forall i,\ b_i \in S \cup S^{-1}$, then letting $u' = b_n^{-1} \cdots b_1^{-1}$, $[w'u'] = [w'][u'] = [w_\emptyset]$, or in other words that $w'u' \leftrightarrow w_\emptyset$. We must show that the claim is also true for $j = n+1$.

Let $w = a_1 \cdots a_{n+1}$ where $\forall i,\ a_i \in S \cup S^{-1}$ and let $u = a_{n+1}^{-1} \cdots a_1^{-1}$. Let $w' = a_2 \cdots a_{n+1}$ and $u' = a_{n+1}^{-1} \cdots a_2^{-1}$. Then since w' is a word of length n, $w'u' \overset{*}{\leftrightarrow} w_\emptyset$ by our inductive hypothesis. However, $w = a_1 w'$ and $u = u'a_1^{-1}$, giving

$$wu = a_1 w' u' a_1^{-1} \overset{*}{\leftrightarrow} a_1 a_1^{-1} \overset{*}{\leftrightarrow} w_\emptyset$$

Thus, $[wu] = [w][u] = [w_\emptyset]$ as desired. A nearly identical argument can be made to show $[u][w] = [w_\emptyset]$ in both the base case and the inductive case, and as such, our claim is proven for all $j \in \mathbb{Z}^+$. □

Having shown that F_n is in fact a group, we will prove a theorem regarding its relationship with a certain subset of $(S \cup S^{-1})^*$ which we define below.

Definition 1.1.14. Let $S = \{s_1, \ldots, s_n\}$ be a set and w a word in $\left\{S \cup S^{-1}\right\}^*$. Then w is said to be a *freely reduced word* if and only if it does not contain a sub-word of the form ss^{-1} or $s^{-1}s$ for any element $s \in S$. Let $\mathcal{W}(S)$ denote the set of all freely reduced words in $(S \cup S^{-1})^*$.

Our goal is to show a $1-1$ correspondence between F_n and $\mathcal{W}(S)$. Before we do so, however, we must define several terms and present several related theorems associated with the field of confluent string rewriting that will be used in our proof.

Definition 1.1.15. Let B be a set and $\Rightarrow$ a binary relation on B. Then the structure $R = (B, \Rightarrow)$ is a *reduction system.*

For the following definitions and theorems (through Theorem 1.1.22), let $R = (B, \Rightarrow)$ be a reduction system.

Definition 1.1.16. We define a relation $\overset{*}{\Rightarrow}$ on B such that $c \overset{*}{\Rightarrow} d$ if and only if $\exists c_1, \ldots, c_n \in S$ such that $c = c_1 \Rightarrow \cdots \Rightarrow c_n = d$. If $c \overset{*}{\Rightarrow} d$, then we say d is a *descendent* of c. If $\forall d,\ c \overset{*}{\not\Rightarrow} d$, then we say c is *irreducible.*

Definition 1.1.17. We say R is *terminating* if and only if there exists no infinite sequence $b_0, b_1, \ldots \in B$ such that $b_0 \Rightarrow b_1 \Rightarrow \cdots$.

Definition 1.1.18. We say R is *locally confluent* if and only if for all $b, c, d \in B$, $b \Rightarrow c$ and $b \Rightarrow d$ implies that there exists $f \in B$ such that $c \overset{*}{\Rightarrow} f$ and $d \overset{*}{\Rightarrow} f$.

Definition 1.1.19. We say R is *confluent* if and only if for all $b, c, d \in B$, $b \overset{*}{\Rightarrow} c$ and $b \overset{*}{\Rightarrow} d$ implies that there exists $f \in B$ such that $c \overset{*}{\Rightarrow} f$ and $d \overset{*}{\Rightarrow} f$.

Theorem 1.1.20. [4, Proposition 1.1.19] R is confluent if and only if it is terminating and locally confluent.

Theorem 1.1.21. [1, Lemma 1.1.10] If R is terminating, $\forall b \in B$, $\exists c \in [b]$ such that c is irreducible.

Theorem 1.1.22. [1, Corollary 1.1.8] If $\mathcal{R}$ is confluent, for all $b \in B$, $[b]$ has at most element that is irreducible.

Consider the reduction system $\mathcal{R} = ((S \cup S^{-1})^*, \to)$. We will prove several lemmas regarding $\mathcal{R}$. First, we will show that $\mathcal{R}$ is terminating. To do so, we introduce the following lemmas.

Lemma 1.1.23. Let $w, v \in (S \cup S^{-1})^*$ with $w \to v$. Then $l(v) = l(w) - 2$.

Proof. Let $w, v \in (S \cup S^{-1})^*$ as above. Then $\exists x, y \in (S \cup S^{-1})^*$ such that $w = x\alpha y$ with $\alpha \in \{ss^{-1}, s^{-1}s\}$ for some $s \in S$. Then $w = x\alpha y \to xy = v$, giving $l(w) = l(x) + l(\alpha) + l(y) = l(x) + l(y) + 2 = l(v) + 2$ and therefore $l(v) = l(w) - 2$ as desired. □

Lemma 1.1.24. Let $w \in (S \cup S^{-1})^*$ such that $\exists w_0, \ldots, w_k \in (S \cup S^{-1})^*$ with $w = w_0 \to \cdots \to w_k$. Then $\forall i,\ l(w_i) = l(w) - 2i$.

Proof. We will employ mathematical induction, inducting on the length of k. For our base case $k = 1$, $w = w_0 \to w_1$. Then by the lemma above, $l(w_1) = l(w) - 2(1)$ as desired.

For our inductive case, assume that the claim is true for $k = n$. Let $w \in (S \cup S^{-1})^*$ for which $\exists w_0, \ldots, w_{n+1} \in (S \cup S^{-1})^*$ such that $w_0 \to \cdots \to w_n \to w_{n+1}$. By our inductive hypothesis, $l(w_n) = l(w) - 2n$. Furthermore, since $w_n \to w_{n+1}$, $l(w_{n+1}) = l(w_n) - 2 = l(w) - 2n - 2 = l(w) - 2(n+1)$ as desired. Therefore, by the principle of mathematical induction, our claim is true for all $k \geq 1$. □

Lemma 1.1.25. $\mathcal{R}$ is terminating.

Proof. Let $w \in (S \cup S^{-1})^*$ with $l(w) = k$. We wish to show that there exists no infinite sequence $w_0, w_1, \ldots \in (S \cup S^{-1})^*$ such that $w = w_0 \to w_1 \to \cdots$. Suppose to the contrary that such an infinite sequence exists. We will consider cases in which k is even and k is odd.

If k is even, consider $l(w_{\frac{k}{2}})$. By the lemma above, $l(w_{\frac{k}{2}}) = l(w) - 2(\frac{k}{2}) = k - k = 0$, giving $w_{\frac{k}{2}} = w_\emptyset$, so $w_{\frac{k}{2}}$ is irreducible, which contradicts the premise that $w_{\frac{k}{2}} \to w_{\frac{k}{2}+1}$. If k is odd, consider $l(w_{\frac{k}{2}})$. By the lemma above, $l(w_{\frac{k-1}{2}}) = l(w) - 2(\frac{k-1}{2}) = k - k + 1 = 1$ so $w_{\frac{k}{2}}$ is irreducible, contradicting the premise that $w_{\frac{k-1}{2}} \to w_{\frac{k-1}{2}+1}$. Therefore, by contradiction, our claim is proven. □

Lemma 1.1.26. $\mathcal{R}$ is confluent.

Proof. By Theorem 1.1.20, since we have just shown above that $\mathcal{R}$ is terminating, it suffices to show that $\mathcal{R}$ is locally confluent. Let $b, c, d \in (S \cup S^{-1})^*$ such that $b \to c$ and $b \to d$. We wish to show that $\exists f \in (S \cup S^{-1})^*$ such that $c \xrightarrow{*} f$ and $d \xrightarrow{*} f$.

As $b \to c$ and $b \to d$, both c and d are obtained by removing different inverse pairs in b. We must consider two cases: (i) the case in which these inverse pairs have no overlap and (ii) the case in which the inverse pairs have overlap.

In case (i), we may write $b = wss^{-1}utt^{-1}v$ for some $w, u, v \in (S \cup S^{-1})^*$ and $s, t \in S \cup S^{-1}$. Without loss of generality, we may consider $b \to c$ and $b \to d$ where $c = wss^{-1}uv$ and $d = wutt^{-1}v$. Note that $c = wss^{-1}uv \to wuv = f$ and $d = wutt^{-1}v \to wuv = f$, giving $c \xrightarrow{*} f$ and $d \xrightarrow{*} f$ as desired.

In case (ii), we may write $b = wss^{-1}sv$ for some $w, v \in (S \cup S^{-1})^*$ and $s \in S \cup S^{-1}$. Without loss of generality, we may consider c obtained by deleting ss^{-1} from b, giving $c = wsv$, and d obtained by deleting $s^{-1}s$ from b, giving wsv. As $c = d$, consider $f = c = d$. Then as $\xrightarrow{*}$ is reflexive, $c \xrightarrow{*} f$ and $d \xrightarrow{*} f$ as desired.

Thus, in all cases, such a word $f \in (S \cup S^{-1})^*$ exists, so we may say that $\mathcal{R}$ is confluent. □

Lemma 1.1.27. Let $w \in (S \cup S^{-1})^*$. Then w contains the sub-word ss^{-1} or $s^{-1}s$ for some $s \in S$ if and only if $\exists v \in (S \cup S^{-1})^*$ such that $w \to v$ and $w \neq v$.

Proof. Let $w \in (S \cup S^{-1})^*$ and assume that w contains a sub-word $\alpha \in \{ss^{-1}, s^{-1}s\}$ for some $s \in S$. Then $\exists x, y \in (S \cup S^{-1})^*$ such that $w = x\alpha y$, giving $w = x\alpha y \to xy \neq w$ as desired.

Now, assume that $\exists v \in (S \cup S^{-1})^*$ such that $w \to v$. Then $\exists x, y \in (S \cup S^{-1})^*$ such that $w = x\alpha y$ with $\alpha \in \{ss^{-1}, s^{-1}s\}$ for some $s \in S$. But this is to say that w contains a sub-word ss^{-1} or $s^{-1}s$ as desired. □

In the corollary below, we simply negate the conditions presented in the lemma above.

Corollary 1.1.28. Let $w \in (S \cup S^{-1})^*$. Then w does not contain a sub-word ss^{-1} or $s^{-1}s$ for any $s \in S$ if and only if $\forall v \in (S \cup S^{-1})^*$, $w \to v$ implies $w = v$. In other words, w is a freely reduced word in $(S \cup S^{-1})^*$ if and only if it is irreducible.

We will use these results from the field of confluent string rewriting to prove the following theorem about free groups.

Theorem 1.1.29. Let $S = \{s_1, \ldots, s_n\}$ be a set. Then there exists a 1-1 correspondence between F_n, the free group of rank n with basis S, and $\mathcal{W}(S)$.

Proof. Let $\phi : \mathcal{W}(S) \to F_n$ such that for $w \in \mathcal{W}(S)$, $\phi(w) = [w] \in F_n$. We wish to show that ϕ is both surjective and injective.

To show surjectivity, select $[w] \in F_n$. We must show that there exists $u \in \mathcal{W}(S)$ such that $\phi(u) = [w]$. Consider our reduction system $\mathcal{R}$. As $\mathcal{R}$ is terminating, by Lemma 1.1.21, $\exists u \in [w]$ such that u is irreducible. By Corollary 1.1.28, u is a freely reduced word in $(S \cup S^{-1})^*$, giving that $u \in \mathcal{W}(S)$ as desired.

To show that ϕ is injective. Let $w_1, w_2 \in \mathcal{W}(S)$ such that $[w_1] = [w_2]$. We must show $w_1 = w_2$. As $w_1, w_2 \in \mathcal{W}(S)$, w_1, w_2 are freely reduced words in $(S \cup S^{-1})^*$, and therefore, by Corollary 1.1.28, w_1, w_2 are irreducible. However, by Lemma 1.1.22, since $\mathcal{R}$ is confluent, $[w_1] = [w_2]$ contains at most one irreducible element. As $w_1, w_2 \in [w_1] = [w_2]$, it must then be the case that $w_1 = w_2$ as desired. $\square$

Having shown that there exists a 1-1 correspondence between F_n, the free group of rank n with basis S and $\mathcal{W}(S)$, the set of freely reduced words in $(S \cup S^{-1})^*$, we may utilize this relationship by referring to an element in F_n as its corresponding freely reduced word in $\mathcal{W}(S)$. This notation will be used in the following section on group presentation.

1.2 Group Presentation

We begin with a definition of the generating set of a group.

Definition 1.2.1. Let G be a group. We say $T \subseteq G$ is a *generating set* of G if and only if $\forall g \in G$, there exists $t_1, t_2, \ldots, t_j$ such that $g = t_1 t_2 \cdots t_j$ and $\forall i$, $t_i \in T$ or $t_i^{-1} \in T$.

In other words, any element in G can be expressed as a product of elements of a generating set T and their inverses.

We begin with this definition because one of the goals of group presentation is to provide a notation with which to write the elements of a group's generating set. A group's presentation also specifies which words in the generators and their inverses reduce to the identity when considered as products in the group. We consider the definition below.

Definition 1.2.2. Let G be a group. Then we say G has the following presentation

$$G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$$

where $S = \{x_1, \ldots, x_n\}$ is a generating set for G and each element r_i is a word in $(S \cup S^{-1})^*$, if and only if $G \cong F_n/N$ where

- F_n is a free group of rank n with basis S
- N is the smallest normal subgroup of F_n such that $\{r_1, \ldots, r_m\} \in N$. In other words, if M is a normal subgroup of F_n containing $\{r_1, \ldots, r_m\}$, then $N \subseteq M$.

The products of generators and their inverses that reduce to the identity in the group, as discussed above, are $r_1, \ldots, r_m$. We call these special words *defining relations.*

To cite an example, consider the following proof regarding the presentation of C_2, the cyclic group of order 2.

Claim 1.2.3. C_2 has the following presentation.

$$C_2 = \langle a \mid a^2 \rangle$$

Proof. Let F_1 be a free group of rank 1 with basis $S = \{a\}$ and let N be the smallest normal subgroup of F_1 containing a^2. We must show that $C_2 \cong F_1/N$. It suffices, however, to show that F_1/N is a cyclic group of order two, since any two cyclic groups of the same order are isomorphic.

Let $A = \{a^{2k} | k \in \mathbb{Z}\}$. We first claim that $A = N$, or in other words that A is the smallest normal subgroup of F_1 containing a^2. By definition, $A \subseteq F_1$. Furthermore, because $a^{2k}a^{2j} = a^{2(j+k)} \in F_1$, A is closed under the operation in

F_1 and because $a^{2k}a^{-2k} = a^{2k-2k} = a^0 = w_\emptyset$, so A is also closed under inverses. Thus, A is a subgroup of F_1. Furthermore, when $k = 1$, we have $a^{2k} = a^2 \in A$. To show that $A \lhd F_1$, let $f = a^z \in F_1$. We must show that $f^{-1}Af \subseteq A$. In other words, $\forall w \in A$, $f^{-1}wf \in A$. Let $w = a^{2k} \in A$. Then

$$f^{-1}wf = a^{-z}a^{2k}a^z = a^{-z+2k+z} = a^{2k}$$

And as $a^{2k} \in A$, we have $f^{-1}wf \in A$ as desired. Thus, $A \lhd F_1$ with $a^2 \in A$, and it remains to show that A is the smallest such normal subgroup of F_1.

Let M be a normal subgroup of F_1 containing a^2. We must show that $A \subseteq M$. Let $w = a^{2k} \in A$. Then $w = a^{2k} = (a^2)^k$, and as M is a group containing a^2, all powers of a^2 must be in M. Thus, $(a^2)^k = a^{2k} = w \in M$, giving $A \subseteq M$ as desired.

We have shown that $A = \{a^{2k} | k \in \mathbb{Z}\}$ is the smallest normal subgroup of F_1 containing a^2, and as such, $A = N$. Next, we claim that F_1/N is a cyclic group of order 2.

Consider $F_1/N = \{wN | w \in F_1\} = \{a^jN | j \in \mathbb{Z}\}$. For any j, j is either odd or even. If j is odd, then $a^jN = a^{2i+1}N = aa^{2i}N = aN$ for some i since $a^{2i} \in N$. Otherwise, if j is even, then $a^jN = a^{2i}N = N$ for some i. Thus, $F_1/N = \{aN, N\}$. Note that $a \in aN$ and $a \notin N$, so $aN \neq N$, and thus $N = \{aN, N\}$ is a group of order 2.

To show that this group is cyclic, note that $\forall g \in F_1/N$, g can be written as a power of aN. Specifically, $(aN)^1 = aN$ and $(aN)^2 = a^2N = N$. Thus, F_1/N is a cyclic group of order 2 and is therefore isomorphic to any other cyclic group of order 2, including C_2. □

Although this proof is sufficient to prove our claim, the proof strategy utilized can only be employed in the special case where the group in question is cyclic. Below, we will provide an alternate proof of Claim 1.2.3 that will employ a more general proof strategy that is more generally applicable beyond this special case. This strategy utilizes a lemma and a theorem that are standard results in group theory. They are presented below.

Lemma 1.2.4. [3, Theorem 2.5.5.] Given G, G' groups and a homomorphism $\phi : G \to G'$, recall $\text{Ker}(\phi) = \{g \in G | \phi(g) = e\}$ where e is the identity in G'. Then $\text{Ker}(\phi) \lhd G$.

Theorem 1.2.5. (First Homomorphism Theorem). [3, Theorem 2.7.1.] Given groups G, G' and a surjective homomorphism $\phi : G \to G'$, then $G' \cong G/\text{Ker}(\phi)$.

We now present an alternate proof of Claim 1.2.3 regarding the presentation of C_2. Note once more that C_2 is the cyclic group of order 2. We will write $C_2 = \{x, e\}$ and define the operation in this group as in the following operation table.

	e	x
e	e	x
x	x	e

Claim 1.2.3 (Alternate Proof). $C_2 = \langle a \mid a^2 \rangle$.

Proof. Let F_1 be a free group of rank 1 with basis $\{a\}$ and let N be the smallest normal subgroup of N containing a^2. We wish to show $C_2 \cong F_1/N$.

We will consider the following map $\psi : F_1 \to C_2$. For $w = a^k \in F_1$, we define $\psi(a^k) = x^k$, where by x^k, we mean the product of k copies of x if k is positive and k copies of x^{-1} if k is negative. To show $C_2 \cong F_1/N$, by the First Homomorphism Theorem, it suffices to show the following.

(1) ψ is a homomorphism.

(2) ψ is surjective.

(3) $\text{Ker}(\psi) = N$.

To show that ψ is a homomorphism, let $w = a^k, u = a^j \in F_1$. We wish to show that $\psi(wu) = \psi(w)\psi(u)$. Note that $wu = a^k a^j = a^{k+j}$. By our definition of ψ above and the definition of exponent notation, we have

$$\psi(wu) = x^{k+j} = x^k x^j = \psi(w)\psi(u)$$

Therefore, ψ is a homomorphism as desired.

To show that ψ is surjective, we simply note that there exist elements in F_1 that ψ maps to each of the two elements in C_2. Namely, $\psi(a^1) = x^1 = x$ and $\psi(a^2) = x^2 = e$. Therefore, ψ is surjective.

To show that $\text{Ker}(\psi) = N$, we must show that $\text{Ker}(\psi)$ is the smallest normal subgroup of F_1 containing a^2. As $\psi(a^2) = x^2 = e$, $a^2 \in \text{Ker}(\psi)$. Furthermore, because ψ is a homomorphism, $\text{Ker}(\psi) \lhd F_1$ by Lemma 1.2.4. As such, it remains to show that $\text{Ker}(\psi)$ is the smallest normal subgroup of F_1 containing a^2.

Let M be a normal subgroup containing a^2. We wish to show that $\text{Ker}(\psi) \subseteq M$. Let $w = a^z \in \text{Ker}(\psi)$. We must show $w \in M$. As $w \in \text{Ker}(\psi)$, $\psi(w) = \psi(a^z) = x^z = e$. If z is odd, $x^z = x \neq e$, so z must be even. As such, $\exists l \in \mathbb{Z}$ with $z = 2l$. So $a^z = a^{2l} = (a^2)^l$. But as M is a group containing a^2, all powers of a^2 must be in M. Thus, $(a^2)^l = a^{2l} = a^z = w \in M$ as desired. This gives $\text{Ker}(\psi) \subseteq M$, and therefore $\text{Ker}(\psi)$ is the smallest normal subgroup of F_1 containing a^2, or in other words that $\text{Ker}(\psi) = N$ as desired. □

This concludes our section on group presentation, a topic that will be revisited after having defined the Lamplighter Group.

1.3 The Infinite Direct Sum

The goal of this section is to define the infinite direct sum and provide examples of groups such a structure defines. As the direct sum forms a group, we must define for it an underlying set and a corresponding operation. We specify these in the following two definitions.

Definition 1.3.1. Let G a group with identity element e. Then the *infinite direct sum of copies of* G, denoted $\oplus_{i=-\infty}^{\infty}(G)_i$, is the group of infinite tuples (x_i), written

$$(x_i) = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$$

where $\forall i$, $x_i \in G$, and such that there exists a finite subset $I = \{i | x_i \neq e\} \subseteq G$.

Definition 1.3.2. Let $(x_i), (y_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$. Then we define $(x_i)(y_i) = (z_i)$ where, $\forall i$, $z_i = x_i y_i$.

Having thoroughly defined the infinite direct sum, we prove that it is, in fact, a group.

Claim 1.3.3. Let G be a group with identity element e. Then $\oplus_{i=-\infty}^{\infty}(G)_i$ is a group.

Proof. Of $\oplus_{i=-\infty}^{\infty}(G)_i$, we must show closure, associativity, the existence of an identity element, and the existence of inverses.

To show closure, we let $(x_i), (y_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$. We must show that $(z_i) = (x_i)(y_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$. We know $\forall i$, $z_i = x_i y_i$. As $x_i, y_i \in G$ and G a group, $x_i y_i \in G$. It remains to show that there are at most finitely many i for which $z_i \neq e$.

Let $A = \{i | x_i \neq e\}$ and $B = \{i | y_i \neq e\}$. As there are at most finitely many i for which $x_i \neq e$ and at most finitely many i for which $y_i \neq e$, both A and B are finite. Consider z_k. If $z_k \neq e$, then either $x_k \neq e$ or $y_k \neq e$, giving $k \in A \cup B$. As both A and B are finite, $A \cup B$ finite. Thus, there are at most $|A \cup B|$ i for which $z_i \neq e$, and therefore $(z_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$ as desired.

To show associativity, let $(x_i), (y_i), (w_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$. We must show $[(x_i)(y_i)](z_i) = (x_i)[(y_i)(z_i)]$.

$$\begin{aligned}
[(x_i)(y_i)](z_i) &= (x_i y_i)(z_i) \\
&= \big((x_i y_i) z_i\big) \\
&= \big(x_i (y_i z_i)\big) \\
&= (x_i)(y_i z_i) \\
&= (x_i)[(y_i)(z_i)]
\end{aligned}$$

Note that $(x_0 y_0)z_0 = x_0(y_0 z_0)$ due to associativity in G, giving associativity in $\oplus_{i=-\infty}^{\infty}(G)_i$ as desired.

To show the existence of an identity element, consider $(e_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$ where $\forall i$, $(e_i) = e$, the identity in G. If we let $(x_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$, then

$$\begin{aligned}(e_i)(x_i) &= (e_i x_i) = (e x_i) = (x_i)\\ (x_i)(e_i) &= (x_i e_i) = (x_i e) = (x_i)\end{aligned}$$

giving the existence of an identity element as desired.

To show the existence of inverses, let $(x_i) \in \oplus_{i=-\infty}^{\infty}(G)_i$ and let $(x_i') \in \oplus_{i=-\infty}^{\infty}(G)_i$ such that $\forall i,\ x_i' = x_i^{-1}$. Then

$$\begin{aligned}(x_i)(x_i') &= (x_i x_i') = (x_i x_i^{-1}) = (e_i)\\ (x_i')(x_i) &= (x_i' x_i) = (x_i \in x_i) = (e_i)\end{aligned}$$

giving the existence of inverses as desired.

Thus, $\oplus_{i=-\infty}^{\infty}(G)_i$ is a group. □

Having proven the above claim in its entirety, we continue to demonstrate the operation amongst elements of an infinite direct sum with an example. Consider $C_3 = \{x, x^2, e\}$, the cyclic group of order 3 with identity element e. Note that operation in this group may be defined as in the following operation table.

	x	x^2	e
x	x^2	e	x
x^2	e	x	x^2
e	x	x^2	e

We may consider the following elements $(a_i), (b_i) \in \oplus_{i=-\infty}^{\infty}(C_3)_i$ where $\forall i$,

$$a_i = \begin{cases} x & \text{if } i = 0 \\ x^2 & \text{if } i = 1 \\ e & \text{otherwise} \end{cases} \qquad b_i = \begin{cases} x & \text{if } i = -1 \\ x^2 & \text{if } i = 0 \\ e & \text{otherwise} \end{cases}$$

We may write

$$\begin{array}{ccccccccccccc} & & & & -3 & -2 & -1 & 0 & 1 & 2 & 3 & & \\ (a_i) & = & (& \cdots & e & e & e & x & x^2 & e & e & \cdots &) \\ (b_i) & = & (& \cdots & e & e & x & x^2 & e & e & e & \cdots &) \end{array}$$

We obtain the element at the i^{th} index of $(z_i) = (a_i)(b_i)$ by simply operating across the i^{th} elements of (a_i) and (b_i) sequentially. For example, we see that $a_0 = x$ and $b_0 = x^2$ as highlighted above. Then $z_0 = a_0 y_0 = xx^2 = x^3 = e$. We compute the rest of this element here.

$$
\begin{array}{rcccccccccccc}
 & & & & {\scriptstyle -3} & {\scriptstyle -2} & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & & \\
(a_i) & = & (& \cdots & e & e & e & x & x^2 & e & e & \cdots &) \\
(b_i) & = & (& \cdots & e & e & x & x^2 & e & e & e & \cdots &) \\
(a_i)(b_i) & = & (& \cdots & e & e & x & e & x^2 & e & e & \cdots &)
\end{array}
$$

In upcoming discussions of the Lamplighter Group, we will utilize the direct sum of copies of $\mathbb{Z}_2$ to define a group to which the Lamplighter Group is isomorphic.

1.4 Properties of The Cayley Graph

The goal of this section is to thoroughly define the Cayley Graph and provide examples of its inner workings as well as further observations one can make regarding such a graph. We begin with the definition of the Cayley Graph below.

Definition 1.4.1. Let G be a group generated by $S = \{s_1, \ldots, s_c\}$. Then the *Cayley Graph* of G with respect to S, denoted $\Gamma_{G,S}$, is the graph with vertex set $V(\Gamma_{G,S})$ and edge set $E(\Gamma_{G,S})$ such that

- $V(\Gamma_{G,S}) = G$
- $E(\Gamma_{G,S}) = \{\ (g, h) \mid \exists s \in S \text{ such that } gs = h\ \}$.

Consider the most basic, non-trivial Cayley Graph, $\Gamma_{C_2,\{x\}}$, the Cayley Graph of the cyclic group of order 2 with generating set $S = \{x\}$ below.

Figure 1.1: $\Gamma_{C_2,\{x\}}$

In accordance with our definition, there is one vertex in the graph for each element of the group. Furthermore, as $xx = x^2 = e$, there is an edge from vertex x to e and, as $ex = x$, there is an edge from vertex e to x. Below, we show several more Cayley Graphs of cyclic groups.

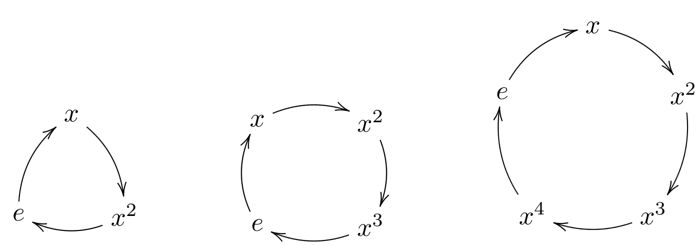


Figure 1.2: Cayley Graphs of several cyclic groups

We continue our discussion below, straying from Cayley Graphs of cyclic groups and focusing on those of groups with multiple generators. For example, consider Figure 1.3, the Cayley Graph of $\mathbb{Z} \times \mathbb{Z}$ under addition with generating set $S = \{(1, 0), (0, 1)\}$. Again, every element of the group is given a vertex, and each vertex is given two outgoing edges to vertices that can be obtained by operating on the initial vertex with a generator $(1,0)$ or $(0,1)$. For example, as $(1,1)(1,0) = (2,1)$, there is an edge from $(1,1)$ to $(2,1)$ and similarly, as $(1,1)(0,1) = (1,2)$, there is an edge from $(1,1)$ to $(1,2)$.

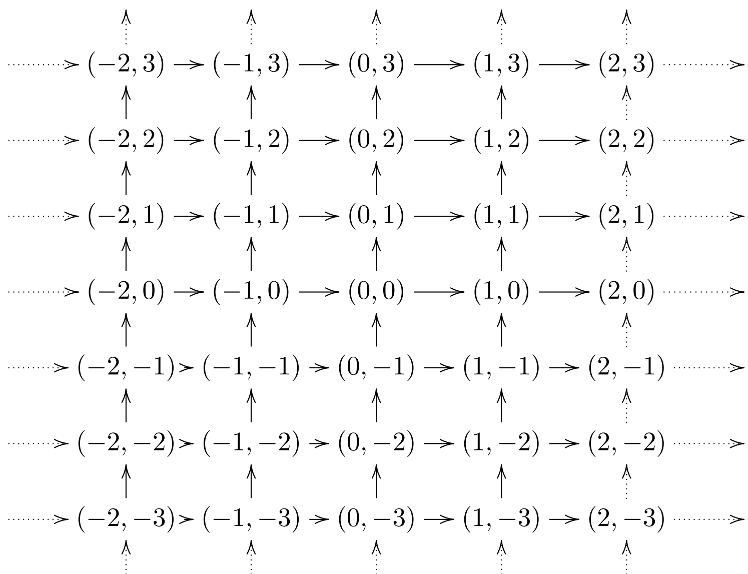


Figure 1.3: $\Gamma_{\mathbb{Z}\times\mathbb{Z},S}$

With this graph in mind, we consider the following definition.

Definition 1.4.2. Let G a group with generating set S and let $g_1, g_k \in G$. Then a path p from g_1 to g_k in $\Gamma_{G,S}$ is a sequence of vertices $p = g_1, \ldots, g_k \in V(\Gamma_{G,S})$ such that $\forall i$ with $1 \leq i \leq k-1$, $\exists s \in S$ such that $g_i s = g_{i+1}$ or $g_i s^{-1} = g_{i+1}$.

For example, the following sequences of vertices are paths from $(-1,-2)$ to $(0,3)$ in $\Gamma_S(\mathbb{Z}\times\mathbb{Z})$.

$$p_1 = (-1,2), (-1,3), (0,3)$$
$$p_2 = (-1,2), (0,2), (0,3)$$
$$p_3 = (-1,2), (0,2), (1,2), (1,3), (0,3)$$

To traverse one of the paths above as an example, consider p_3. We know that $(-1,2)(1,0) = (0,2)$, $(0,2)(1,0) = (1,2)$, $(1,2)(0,1) = (1,3)$ and $(1,3)(1,0)^{-1} = (0,3)$, and thus, p_1 is a path in $\Gamma_S(\mathbb{Z}\times\mathbb{Z})$.

Definitionally, each successive vertex in a path is obtained by operating on that vertex with either a generator or its inverse. Thus, we may associate these generators (and their inverses) with the path itself, which motivates the following definition.

Definition 1.4.3. Let G a group with generating set S and let $p = g_1, \ldots, g_k$ a path in $\Gamma_{G,S}$. Then the *word associated with path* p is the word $w = s_1 \cdots s_{k-1}$ in $(S \cup S^{-1})^*$ where $\forall i$ with $1 \leq i \leq k-1$

$$s_i = \begin{cases} s & \text{if} \quad g_i s = g_{i+1} \\ s^{-1} & \text{if} \quad g_i s^{-1} = g_{i+1} \end{cases}$$

Furthermore, we define the length of the path p, denoted $l(p)$, to be the length of its associated word.

Continuing with our example above, if we let $a = (1,0)$ and $b = (0,1)$, then we may write the words corresponding to p_1, p_2, p_3 as follows:

$p_1 = ((-1,2), (-1,3), (0,3))$
Corresponding word: $w_1 = ba$

$p_2 = ((-1,2), (0,2), (0,3))$
Corresponding word: $w_2 = ab$

$p_3 = ((-1,2), (0,2), (1,2), (1,3), (0,3)$
Corresponding word: $w_3 = aaba^{-1}$

Note that if we consider any of these words w_i as products, then $(-1,2)w_i = (0,3)$.

Certainly, some words associated with paths from $(-1,2)$ to $(0,3)$ in $\Gamma_{\mathbb{Z}\times\mathbb{Z},S}$ are longer than others, and as such, so are the paths themselves. We take this into account in the following definition.

Definition 1.4.4. Let $g, h \in G$, a group with generating set S. Then the *distance* from g to h in G, denoted $d(g,h)$ is the length of the shortest word w in $(S \cup S^{-1})^*$ such that $gw = h \in G$. In other words, $d(g,h)$ is the length of the shortest path from g to h in $\Gamma_{G,S}$.

Of the three words recorded above, certainly w_1 and w_2, both of length 2, are the shortest. It is in fact true that, the shortest word w_i such that $(-1,2)w_i = (0,3)$ is of length 2, giving $d((-1,2),(0,3)) = 2$, though we will not prove this result here. Next, we define a special distance.

Definition 1.4.5. Let $g \in G$, a group with generating set S. Then the *word length* of g, denoted $l(g) = d(e,g)$.

Having thoroughly defined the notion of distance and word length, we next consider the following definition.

Definition 1.4.6. Let $g \in G$, a group with generating set S. Then g is a *dead-end element* of G if and only if $\forall s \in S$, $l(gs) \leq l(g)$ and $l(gs^{-1}) \leq l(g)$.

To illustrate an example of a dead-end element, we first consider the Cayley Graph of $\mathbb{Z}$ under addition, which models the standard number line.

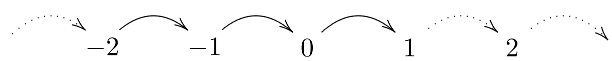


Figure 1.4: $\Gamma_{\mathbb{Z},\{1\}}$

Note that 1, when operated on by the generator 1, gives us 2, and as $l(1) = 1$ and $l(2) = 2$, $l(1) < l(2)$. Thus, 1 is not a dead-end element in $\mathbb{Z}$ if we consider this generating set. However, complexities may arise when considering $\mathbb{Z}$ with a different generating set. For example, we select $\{2, 3\}$ to be our generating set, appealing to the discussion above and thus resulting in a more complex Cayley Graph, as shown below.

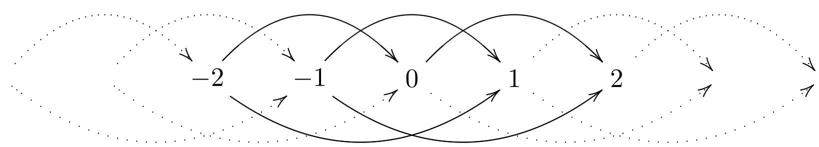


Figure 1.5: $\Gamma_{\mathbb{Z},\{2,3\}}$

Again, we consider 1, which we claim to be a dead-end element in $\mathbb{Z}$ (as generated by $\{2, 3\}$). There exists a path p of length 2 from the identity to 1 (namely $p = 0, 3, 1$, since $0 + 3 = 3$ and $3 - 2 = 1$). However, since $1, -1 \notin S$, $l(1) > 1$, this path of length 2 must be the shortest path, giving $l(1) = 2$.

To show 1 is a dead-end element, we must consider the length of elements obtained when operating 1 by the generators and their inverses. The lengths of these elements are listed below.

$$l(1 + 2) = l(3) \qquad l(1 + 3) = l(4) \qquad l(1 - 2) = l(-1) \qquad l(1 - 3) = l(-2)$$

As $3 \in S$ and $-2 \in S^{-1}$, $l(3) = l(2) = 1$, giving $l(3) = l(2) \leq l(1)$. Furthermore, there exists a path of length 2 from 0 to 4 (since $2 + 2 = 4$) and there exists a path of length 2 from 0 to -1 (since $2 - 3 = -1$), giving $l(4), l(-1) \leq 2 = l(1)$.

Thus, $\forall s \in \{2,3\}$, $l(1+s) \leq l(1)$ and $l(1+s^{-1}) \leq l(1)$, so 1 is a dead-end element.

When traversing the Cayley Graph out from the identity, it might seem intuitive to expect length of each element to increase with each edge traversed. In the case of dead-end elements, however, continuing does not result in increased length. This is to say that any element of distance 1 from a dead-end element is contained in the set defined below.

Definition 1.4.7. Let G be a group with generating set S. Then the *ball of radius* k, denoted $B_S(k)$, is the following set.

$$B_S(k) = \{g \in G | l(g) \leq k\}$$

As 1 is a dead-end element in G with length 2, any element of distance 1 from 1 has length less than or equal to that of 1 and therefore is a member of $B_S(2)$. In fact, $B_S(2) = \{-6, \dots, 6\}$, as each of these elements has length less than or equal to 2.

Given a dead-end element, if we are to find a path to an element with length greater than it, we must exit its ball entirely. The length of the shortest path from a dead-end element to an element outside its ball might be greater or less than similar paths from other dead-end elements of equal length. Such a distinction motivates the following definition.

Definition 1.4.8. Let g be a dead-end element in G, a group with generating set S. Then the *depth* of g is equal to $\min\{d(g,h) | h \in G, h \notin B_S(l(g))\}$.

Note that if g is a dead-end element in G as generated by S, then the depth of g is at least 2, since the elements reachable via paths of length 1 from g are all of the form gs or gs^{-1} for some $s \in S$, giving $gs, gs^{-1} \in B_S(l(g))$.

As an example, we may compute the depth of 1 in $\Gamma_{\mathbb{Z},\{2,3\}}$. As $l(1) = 2$, we must find the shortest path from 1 to an element not contained in $B_S(2) = \{-6, \dots, 6\}$. Consider 7. As $7 = 0 + 3 + 2 + 2$, $l(7) = 3$, giving $l(7) \notin B_S(2)$. Furthermore, as $1+3+3 = 7$, $d(1,7) = 2$, giving that 1 is of depth 2 in $\Gamma_{\mathbb{Z},\{2,3\}}$.

Later discussions will identify within the Lamplighter Group dead-end elements of arbitrary depth. That is, for all n there exists an element in the Lamplighter Group with depth at least n. Having concluded our Chapter of Preliminaries, we begin by defining the Lamplighter Group in the opening section of our next chapter.

Chapter 2

The Lamplighter Group

2.1 Motivation

We begin our discussion by defining a dynamical system. For our purposes, we consider a system with a static object and several types of modifications that can be dynamically performed upon this object.

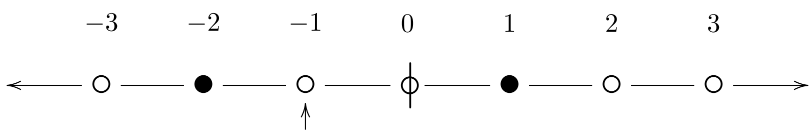
For our static object, consider a road of infinite length lined with streetlamps in either direction, indexed by the integers (a structure we will call the "lampstand"). A finite number of lamps along the lampstand are illuminated while all others remain off, and a lamplighter stands at one lamp. We may dynamically modify this object by allowing the lamplighter to walk in either direction and change the state of the lamp at which is standing.

This system serves as a visual model for several groups, each of which are the same up to isomorphism. This group, to which we refer as the Lamplighter Group, will be defined later on. First, we will give rigor to our system as expressed above with the following definitions.

Definition 2.1.1. Let $\mathcal{L}$ denote the set of all possible configurations as described above.

An arbitrary configuration in $\mathcal{L}$ could easily be represented pictorially. For example, let $c_0 \in \mathcal{L}$ be the configuration with lamps at indices -2 and 1 illuminated and the lamplighter positioned at lamp at index -1. We show c_0 below, where a closed circle ($\bullet$) represents an illuminated bulb, an open circle ($\circ$) represents an unilluminated one, and the position of the lamplighter is indicated by an arrow pointing at the index corresponding to the lamp.

Definition 2.1.2. Let $\varepsilon \in \mathcal{L}$ be the configuration in which all lamps are unilluminated and the lamplighter stands at index 0. We will refer to this configuration as the "empty lampstand".



The dynamics of the system are meant to mimic the movements of the lamplighter about the lampstand and the changes he makes to the state of individual lamps. Atomically, we may consider these dynamics as the following three tasks that the lamplighter may perform on any arbitrary configuration in $\mathcal{L}$.

(1) Switch on/off the lamp at which the he currently stands.

(2) Move one lamp to his right.

(3) Move one lamp to his left.

Note that by performing tasks (2) or (3) $k \in \mathbb{Z}$ times, the lamplighter achieves the task of having moved k lamps to either the right or left respectively. These dynamics are defined below.

Definition 2.1.3. Let $\mathcal{T} = \{\alpha, \tau, \tau'\}$ where every element in $\mathcal{T}$ is a function from $\mathcal{L}$ to itself such that, given $l \in \mathcal{L}$, we define

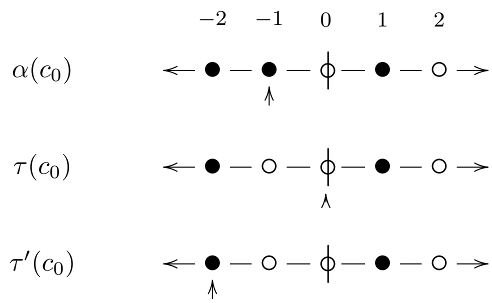
- $\alpha(l)$ is the configuration resulting from having performed task (1) on l.
- $\tau(l)$ is the configuration resulting from having performed task (2) on l.
- $\tau'(l)$ is the configuration resulting from having performed task (3) on l.

As applying a function in $\mathcal{T}$ to a configuration in $\mathcal{L}$ is equivalent to performing its corresponding task on this configuration, we will use these terms interchangeably.

Revisiting our example configuration c_0, we show $\alpha(c_0)$, $\tau(c_0)$, and $\tau'(c_0)$ below:

Note that we may obtain any configuration in $\mathcal{L}$ by applying a finite sequence of tasks to ε, the empty lampstand. Again, we revisit our example configuration c_0, the configuration with lamps at indices -2 and 1 illuminated and the lamplighter positioned at lamp at index -1. We attain this configuration using the following sequence of tasks:

- Move the lamplighter two lamps to his left (to index -2).
- Switch on/off the lamp at which the lamplighter stands (index -2 on).



- Move the lamplighter three lamps to his right (to index 1).
- Switch on/off the lamp at which the lamplighter stands (index 1 on).
- Move the lamplighter two lamps to his left (to index -1).

Such a sequence is not unique. We could just as easily use this sequence of tasks to achieve the same end.

- Move the lamplighter one lamp to his right (to index 1).
- Switch on/off the lamp at which the lamplighter stands (index 1 on).
- Move the lamplighter three lamps to his left (to index -2).
- Switch on/off the lamp at which the lamplighter stands (index -2 on).
- Move the lamplighter one lamps to his right (to index -1).

This motivates the following theorem.

Theorem 2.1.4. For all $l \in \mathcal{L}$, there exists a finite sequence $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that $\forall i$, $\gamma_i \in \mathcal{T}$ and $\gamma_k(\gamma_{k-1}(\cdots(\gamma_1(\varepsilon)))) = l$.

Proof. Let $c \in \mathcal{L}$ where lamps at indices $b_1, \ldots, b_n$ are illuminated and the lamplighter is positioned at index z. Then

$$c = \gamma^z(\alpha_{b_n}(\alpha_{b_{n-1}}(\cdots(\alpha_{b_1}(\varepsilon)))))$$

where for $l_* \in \mathcal{L}$

$$\alpha_j(l_*) = \begin{cases} \tau'^j(\alpha(\tau^j(l_*))) & \text{if } j \geq 0 \\ \tau^{-j}(\alpha(\tau'^{-j}(l_*))) & \text{if } j < 0 \end{cases}$$

and

$$\gamma^z = \begin{cases} \tau^z & \text{if } z \geq 0 \\ \tau'^{-z} & \text{if } z < 0 \end{cases}$$

□

We may use the language of traditional group theory to illustrate this dynamic system as a series of groups, each of which are the same up to isomorphism. The first of these groups is presented below.

2.1.1 Integer Subset Definition

We wish to define a group L_2, to which we will refer as the Lamplighter Group. We define the underlying set and operation in this group in the following two definitions.

Definition 2.1.5. $L_2 = \{(S, z) \mid S \subseteq \mathbb{Z} \text{ (finite) and } z \in \mathbb{Z}\}$.

Definition 2.1.6. Let $l_1, l_2 \in L_2$ with $l_1 = (S, x)$ and $l_2 = (T, y)$. Then

$$l_2 l_1 = \big((S \cup T') - (S \cap T'), x + y\big)$$

where $T' = \{t + x \mid t \in T\}$.

Having defined this set and its corresponding operation, we go on to show that it is, in fact, a group.

Claim 2.1.7. L_2 is a group.

Proof. To prove that L_2 is a group, we must show closure, associativity, the existence of an identity element, and the existence of inverses.

To show closure, let $l_1, l_2 \in L_2$ as above. We wish to show $l_1 l_2 \in L_2$. By the definition of operation in L_2, $l_2 l_1 = \big((S \cup T') - (S \cap T'), x + y\big)$. Note that $(S \cup T') - (S \cap T') \subseteq \mathbb{Z}$ and finite since S and T' are both subsets of $\mathbb{Z}$ and finite. Furthermore, $x + y \in \mathbb{Z}$, and so $l_2 l_1 \in L_2$.

To show associativity, let $l_1, l_2, l_3 \in L_2$ with $l_1 = (S, x)$, $l_2 = (T, y)$, $l_3 = (R, z)$. Then

$$\begin{aligned} l_2 l_1 &= (S \cup T' - S \cap T', x + y) \text{ where } T' = \{t + x | t \in T\} \\ l_3 l_2 &= (T \cup R' - T \cap R', y + z) \text{ where } R' = \{r + y | r \in R\} \end{aligned}$$

and furthermore

$$\begin{aligned} l_3(l_2 l_1) &= \big((S \cup T' - S \cap T') \cup R'' - (S \cup T' - S \cap T') \cap R'', x + y + z\big) \\ &\qquad \text{where } R'' = \{r + x + y | r \in R\} = \{r + x | r \in R'\} \\ (l_3 l_2) l_1 &= (S \cup A - S \cap A, x + y + z) \\ &\qquad \text{where } A = \{x + \alpha | \alpha \in T \cup R' - T \cap R'\} \end{aligned}$$

We wish to show $l_3(l_2 l_1) = (l_3 l_2) l_1$. As the second entry in each ordered pair above is $x + y + z$, it suffices to show equality amongst the first entries. In other words, we must show

$$(S \cup T' - S \cap T') \cup R'' - (S \cup T' - S \cap T') \cap R'' = S \cup A - S \cap A$$

We will denote the set referenced on the left-hand-side of this equality by LHS and the set on the right-hand-side by RHS.

First, we will show that $LHS \subseteq RHS$. Let $k \in LHS$. We must show that $k \in RHS$. As $k \in LHS$, we will examine each of the following cases:

(i) $k \in S,\ k \notin T',\ k \notin R''$

(ii) $k \in T',\ k \notin S,\ k \notin R''$

(iii) $k \in R'',\ k \notin S,\ k \notin T'$

Consider case (i), where $k \in S,\ k \notin T',\ k \notin R''$. As $k \in S$, to show $k \in RHS$, it suffices to show $k \notin A$.

- Because $k \notin T'$, $\forall t \in T$, $t + x \neq k$.
- Because $k \notin R''$, $\forall r \in R'$, $r + x \neq k$.
- Thus, $\forall \beta \in T \cup R'$, $\beta + x \neq k$.
- Furthermore, $\forall \alpha \in T \cup R' - T \cap R'$, $\alpha + x \neq k$, giving $k \notin A$.

Consider case (ii), where $k \in T',\ k \notin S,\ k \notin R''$. As $k \notin S$, to show $k \in RHS$, it suffices to show $k \in A$.

- Because $k \in T'$, $\exists t \in T$ with $t + x = k$, giving $t = k - x$.
- Because $k \notin R''$, $\forall r \in R$, $r + x + y \neq k$, giving $r + y \neq k - x$.
- As $r + y \neq k - x\ \forall r \in R$, $k - x \notin R'$, giving $t \notin R'$.
- Because $t \in T$ and $t \notin R'$, we have $t \in T \cup R' - T \cap R'$.
- Thus, $t + x = k \in A$.

Consider case (iii), where $k \in R'',\ k \notin S,\ k \notin T'$. Again, since $k \notin S$, it suffices to show $k \in A$.

- Because $k \in R''$, $\exists r \in R$ with $r + x + y = k$, giving $r + y = k - x$.
- Because $r \in R$, $r + y = k - x \in R'$.
- Because $k \notin T'$, $\forall t \in T$, $t + x \neq k$, giving $t \neq k - x$, so $k - x \notin T$.
- Since $k - x \in T$ and $k - x \in R'$, we have $k - x \in T \cup R' - T \cap R'$.
- Thus $(k - x) + x = k \in A$.

As all three cases result in $k \in RHS$, we have shown $LHS \subseteq RHS$ as desired.

Next, we will show $RHS \subseteq LHS$. Now, let $k \in RHS$. We must show that $k \in LHS$. As $k \in RHS$, we will examine two cases

(i) $k \in S,\ k \notin A$

(ii) $k \notin S,\ k \in A$

Consider case (i), where $k \in S,\ k \notin A$. As $k \notin A$, we have $k - x \notin T \cup R' - T \cap R'$. Thus, we examine two sub-cases:

(a) $k - x \in T,\ k - x \in R'$

(b) $k - x \notin T,\ k - x \notin R'$

In sub-case (a), we have $k - x \in T,\ k - x \in R'$.

- Because $k - x \in T,\ (k - x) + x = k \in T'$.
- Because $k - x \in R'$, $(k - x) + x = k \in R''$.
- Because $k \in S,\ k \in T'$, $k \notin S \cup T' - S \cap T'$.
- Because $k \in R''$, $k \notin S \cup T' - S \cap T'$, $k \in LHS$.

In sub-case (b), we have $k - x \notin T,\ k - x \notin R'$.

- Because $k - x \notin T,\ (k - x) + x = k \notin T'$.
- Because $k - x \notin R'$, $(k - x) + x = k \notin R''$.
- Because $k \in S,\ k \notin T'$, $k \in S \cup T' - S \cap T'$.
- Because $k \notin R''$, $k \in S \cup T' - S \cap T'$, $k \in LHS$.

Thus, for case (i), we have $k \in LHS$.

Consider case (ii), where $k \notin S,\ k \in A$. As $k \in A$, we have $k = x \in T \cup R' - T \cap R'$. Again, we examine two sub-cases:

(a) $k - x \in T,\ k - x \notin R'$

(b) $k - x \in R',\ k - x \notin T$

In sub-case (a), we have $k - x \in T,\ k - x \notin R'$.

- Because $k - x \in T,\ (k - x) + x = k \in T'$.
- Because $k - x \notin R'$, $(k - x) + x = k \notin R''$.
- Because $k \notin S,\ k \in T'$, $k \in S \cup T' - S \cap T'$.
- Because $k \notin R''$, $k \in S \cup T' - S \cap T'$, $k \in LHS$.

In sub-case (b), we have $k - x \in R',\ k - x \notin T$.

- Because $k - x \notin T,\ (k - x) + x = k \notin T'$.
- Because $k - x \in R'$, $(k - x) + x = k \in R''$.

- Because $k \notin S$, $k \notin T'$, $k \notin S \cup T' - S \cap T'$.
- Because $k \in R''$, $k \notin S \cup T' - S \cap T'$, $k \in LHS$.

Thus, for case (ii), we have $k \in LHS$ as well, giving $RHS \subseteq LHS$ as desired.

To show the existence of an identity element, let $l_1 = (S, z)$ and $l_0 = (\emptyset, 0)$. We must show $l_0 l_1 = l_1 l_0 = l_1$. Observe the following:

$$\begin{aligned} l_0 l_1 &= (\emptyset \cup S' - \emptyset \cap S', x + 0) \text{ where } S' = \{0 + s \mid s \in S\} = S \\ &= (\emptyset \cup S - \emptyset \cap S, x) \\ &= (S, x) = l_1. \end{aligned}$$

$$\begin{aligned} l_1 l_0 &= (S \cup E' - S \cap E', x + 0) \text{ where } E' = \{x + \varepsilon \mid \varepsilon \in \emptyset\} = \emptyset \\ &= (S \cup \emptyset - S \cap \emptyset, x) \\ &= (S, x) = l_1. \end{aligned}$$

Thus, l_0 is an identity in L_2, and as such, we will refer to $l_0 = (\emptyset, 0)$ as e.

To show the existence of inverses in L_2, let $l = (S, z)$ and let $l' = (S', -z)$ where $S' = \{s - z \mid s \in S\}$. We must show $ll' = l'l = (\emptyset, 0)$. Observe the following.

$$\begin{aligned} ll' &= (S' \cup S^0 - S' \cap S^0, -z + z) \text{ where } S^0 = \{s - z \mid s \in S\} \\ &\qquad \text{Note that} S^0 = S'. \\ &= (S' \cup S' - S' \cap S', 0) \\ &= (\emptyset, 0) = e \end{aligned}$$

$$\begin{aligned} l'l &= (S \cup S'' - S \cap S'', z - z) \text{ where } S'' = \{s + z \mid s \in S'\} \\ &\qquad \text{Note that} S'' = \{(s - z) + z | s \in S\} = \{s | s \in S\} = S \\ &= (S \cup S - S \cap S, 0) \\ &= (\emptyset, 0) = e \end{aligned}$$

Thus, for any element $l \in L_2$, l' is proven to be the inverse of l in L_2, and as such, we will refer to l' as l^{-1}. □

Thus, we have proven that L_2 is a group. We now identify specific elements in L_2 to which we will refer later on.

Definition 2.1.8. Let $T_2 = \{a, t, a^{-1}, t^{-1}\}$ where $a = (\{0\}, 0)$ and $t = (\emptyset, 1)$.

Claim 2.1.9. $a^{-1} = a$.

Proof. We must show that $a^2 = e$.

$$a^2 = (\{1\} \cup \{1\} - \{1\} \cap \{1\}, 0 + 0) = (\{1\} - \{1\}, 0) = (\emptyset, 0) = e.$$

□

Claim 2.1.10. $t^{-1} = (\emptyset, -1)$.

Proof. Let $t' = (\emptyset, -1)$. We wish to show $tt' = t't = e$.

$$tt' = (\emptyset \cup \emptyset - \emptyset \cap \emptyset, -1 + 1) = (\emptyset, 0) = e$$

$$t't = (\emptyset \cup \emptyset - \emptyset \cap \emptyset, 1 - 1) = (\emptyset, 0) = e$$

□

Thus, $T_2 = \{a, t, t^{-1}\}$.

2.1.2 Connection

As mentioned, the Lamplighter Group L_2 models the dynamics presented in the system we defined earlier. In this section, we examine the connection between the dynamic system and the group L_2, our goal being to prove the following theorem.

Theorem 2.1.11. For all $l \in L_2$, l can be expressed as a product of a, t, t^{-1}.

We begin by showing a 1-1 correspondence between L_2, our traditionally defined group, and $\mathcal{L}$, the set of configurations in our dynamic system.

Definition 2.1.12. Let $l \in \mathcal{L}$ be the configuration in which the bulbs at indices in $S = \{s_1, \ldots, s_k\}$ are illuminated and the lamplighter is positioned at x. Then we define a function $f : \mathcal{L} \to L_2$ such that $f(l) = (S, x)$.

Note that to every element in L_2, f maps some configuration in $\mathcal{L}$, and furthermore that every configuration in $\mathcal{L}$ can be mapped to only one unique element in L_2. Thus, f establishes a 1-1 correspondence between $\mathcal{L}$ and L_2. Next, we show another 1-1 correspondence, this one between $T_2 = \{a, t, t^{-1}\}$, a subset of L_2, and $\mathcal{T} = \{\alpha, \tau, \tau'\}$, our set of tasks. We record this 1-1 correspondence below.

Definition 2.1.13. Let $g : \mathcal{T} \to T_2$ such that

$$g(\alpha) = a \qquad g(\tau) = t \qquad g(\tau') = t^{-1}$$

Note that g is a 1-1 correspondence by its construction.

Keeping in mind that our final goal is to prove Theorem 2.1.11, this proof will require use of the technical lemma proven below.

Lemma 2.1.14. Let $c \in \mathcal{L}$, $\gamma \in \mathcal{T}$. Then

$$g(\gamma) \cdot f(c) = f(\gamma(c))$$

Proof. Let $c \in \mathcal{L}$ be the configuration in which the bulbs at indices $s_1, \ldots, s_k$ are illuminated and the lamplighter is standing at position x. Note that $f(c) = (S, x)$ where $S = \{s_1, \ldots, s_k\}$. We must consider three possible cases:

(i) $\gamma = \alpha$.
We compute $g(\alpha) \cdot f(c)$ as follows:

$$g(\alpha) \cdot f(c) = a \cdot f(c) = (\{0\}, 0)(S, x) = (S \cup \{x\} - S \cap \{x\}, x) = f(c')$$

where c' is the configuration where the bulb at index x is turned on if the bulb at index x is turned off in configuration c, and the bulb at x is off if the bulb at x is on in configuration c. Such a configuration can be acquired by performing task (1) on configuration c, and as such, $c' = \alpha(c)$, giving $f(c') = f(\alpha(c)$, and furthermore that, as desired,

$$g(\alpha) \cdot f(c) = f(\alpha(c)).$$

(ii) $\gamma = \tau$.
We compute $g(\tau) \cdot f(c)$ as follows:

$$g(\tau) \cdot f(c) = t \cdot f(c) = (\emptyset, 1)(S, x) = (S, x + 1) = f(c')$$

where c' is the configuration where the lamplighter stands one index to the right from the index at which he stands in configuration c. Such a configuration can be acquired by performing task (2) on configuration c, and as such, $c' = \tau(c)$, giving $f(c') = f(\tau(c))$, and furthermore that, as desired,

$$g(\tau) \cdot f(c) = f(\tau(c)).$$

(iii) $\gamma = \tau'$.
We compute $g(\tau') \cdot f(c)$ as follows:

$$g(\tau') \cdot f(c) = t^{-1} \cdot f(c) = (\emptyset, -1)(S, x) = (S, x - 1) = f(c')$$

where c' is the configuration where the lamplighter stands one index to the left from the index at which he stands in configuration c. Such a configuration can be acquired by performing task (3) on configuration c, and as such, $c' = \tau'(c)$, giving $f(c') = f(\tau'(c))$, and furthermore that, as desired,

$$g(\tau') \cdot f(c) = f(\tau'(c)).$$

□

Having proven the claim above, note the special case when $c = \varepsilon$, the empty lampstand. In such a case, for $\gamma \in \mathcal{T}$, we have $g(\gamma) \cdot f(\varepsilon) = f(\gamma(\varepsilon))$. We know, however, that $f(\varepsilon) = e$, the identity in L_2. Thus, this gives us

$$g(\gamma) = f(\gamma(\varepsilon)).$$

With these remarks in mind, we prove the following theorem, which is the goal of this section.

Theorem 2.1.11. For all $l \in L_2$, l can be expressed as a product of a, t, t^{-1}.

Proof. Let $l \in L_2$. Then as f is onto L_2, there exists some configuration $c \in \mathcal{L}$ such that $f(c) = l$. By Claim 2.1.4, we know that there exists $\gamma_1 \ldots, \gamma_k$ such that $c = \gamma_k(\cdots(\gamma_1(\varepsilon)))$. We will employ the principle of mathematical induction, inducting on the value of k.

For our base case, let $k = 1$. Then $c = \gamma_1(\varepsilon)$. This gives $f(c) = f(\gamma_1(\varepsilon))$. However, by Lemma 2.1.14,

$$f(\gamma_1(\varepsilon)) = g(\gamma_1) \cdot f(\varepsilon) = g(\gamma_1) \cdot e = g(\gamma_1)$$

By the definition of g, $g(\gamma_1) \in T_2 = \{a, t, t^{-1}\}$, proving our base case.

For our inductive case, assume that the claim is true for $k = n$. We wish to show that it is also true for $k = n + 1$. Let $c = \gamma_{n+1}(\cdots(\gamma_1(\varepsilon)))$. Then $c = \gamma_{n+1}(b)$ where $b = \gamma_n(\cdots(\gamma_1(\varepsilon)))$. However, by Lemma 2.1.14,

$$f(c) = f(\gamma_{n+1}(b)) = g(\gamma_{n+1}) \cdot f(b)$$

By our inductive hypothesis, as $f(b) = g(\gamma_n) \cdots g(\gamma_1)$. Thus

$$f(c) = g(\gamma_{n+1}) \cdot g(\gamma_n) \cdots g(\gamma_1)$$

Again, by the definition of g, $\forall i,\ g(\gamma_i) \in T_2$, proving our inductive case.

Therefore, by mathematical induction, our claim is proven. □

Thus, using the language of geometric group theory, we have shown that L_2 is generated by T_2 by exploiting L_2's connection to the dynamic system whose discussion opened this chapter. However, we may also prove the same claim using traditional group theory.

Theorem 2.1.11. (Alternate Proof) For all $l \in L_2$, l can be expressed as a product of a, t, t^{-1}.

Proof. Let $l = (S, x) \in L_2$ where $S = \{s_1, \ldots, s_k\}$ and let $l' = t^x \left(\prod_{s \in S} t^{-s} a t^s\right)$. We claim that $l = l'$.

To show this result, we first note the following

- For $k \in \mathbb{Z}$, $t^{-k}at^k = (\{k\}, 0)$.
  This is true because $t^k = (\emptyset, k)$ and $t^{-k} = (\emptyset, -k)$, and therefore

$$t^{-k}at^k = (\emptyset, -k)(\{0\}, 0)(\emptyset, k) = (\{k\}, -k)(\emptyset, k) = (\{k\}, 0).$$

- For $j, k \in \mathbb{Z}$, $(\{j\}, 0)(\{k\}, 0) = (\{j, k\}, 0)$.
- Finally, $t^z = (\emptyset, z)$.

With this in mind, we consider the following.

$$\begin{aligned} l' = t^x \left(\prod_{s \in S} t^{-s}at^s \right) &= t^x \big[(t^{-s_k}at^{s_k})(t^{-s_{k-1}}at^{s_{k-1}}) \cdots (t^{-s_1}at^{s_1}) \big] \\ &= (\emptyset, x) \big[(\{s_k\}, 0)(\{s_{k-1}\}, 0) \cdots (\{s_1\}, 0) \big] \\ &= (\emptyset, x)(\{s_1, \ldots, s_k\}, 0) \\ &= (\{s_1, \ldots, s_k\}, x) = (S, x) = l \end{aligned}$$

□

Thus, $\{a, t\}$ is a generating set of L_2. Here, we claim that it is, in fact, the smallest such generating set.

Theorem 2.1.15. $\{a, t\}$ is the smallest generating set of L_2.

Proof. Suppose to the contrary that there exists a generating set R of L_2 with $|R| \leq 2$. Then $|R| = 1$ and we may write $R = \{r\}$ for some $r \in L_2$.

As $\{r\}$ is a generating set of L_2, $\forall l \in L_2$, $l = r^z$ or $l = (r^{-1})^z = r^{-z}$ for some $z \in \mathbb{Z}^+ \cup \{0\}$. Thus, $\exists p, q \in \mathbb{Z}$ such that $a = r^p$, $t = r^q$. Then

$$at = r^p r^q = r^{p+q} = r^{q+p} = r^q r^p = ta$$

But we know $at = (\{1\}, 1)$ and $ta = (\{0\}, 1)$, giving $at \neq ta$, a contradiction. Thus, no such generating set of cardinality 1 may exist, proving our claim. □

In this opening sub-section, we have discussed how L_2 models the dynamics of the system of configurations in $\mathcal{L}$. We will specify two more groups that also model the dynamics of this system, the first of which appears below.

2.1.3 Infinite Sum Definition

Here, we define another group ${L_2}'$ that models our dynamic system, eventually proving that ${L_2}' \cong L_2$. We begin by defining the underlying set and operation of this group in the following two definitions below.

Definition 2.1.16. Let $L_2{}'$ be the following set:

$$L_2{}' = \{\big((x_i), z\big) \mid x \in \oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i, z \in \mathbb{Z}\}.$$

where $\oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i$ is the infinite direct sum of copies of $\mathbb{Z}_2$. As such, (x_i) is an infinite tuple in which each integer index is assigned a binary value of 0 or 1. Note that only finitely many entries in (x_i) may hold a value of 1.

Definition 2.1.17. Let $l_1, l_2 \in L_2{}'$ with $l_1 = \big((x_i), n\big)$ and $l_2 = \big((y_i), m\big)$. Then

$$l_2 l_1 = \big((z_i), n+m\big)$$

where $\forall i,\ z_i = x_i + y_{i-n}$.

Having defined $L_2{}'$, we show that it is a group below.

Claim 2.1.18. $L_2{}'$ is a group.

Proof. To show that $L_2{}'$ is a group, we must show closure, associativity, the existence of an identity element, and the existence of inverses.

To show closure, let $l_1 = \big((x_i), n\big),\ l_2 = \big((y_i), m\big) \in L_2{}'$. We must show $l_2 l_1 \in L_2{}'$. By the definition of operation in $L_2{}'$, $l_2 l_1 = \big((z_i), n+m\big)$ where $\forall i,\ z_i = x_i + y_{i-n}$. Note, however, that as $\oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i$ is closed, $(z_i) \in \oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i$, and furthermore, as $\mathbb{Z}$ is closed, $n+m \in \mathbb{Z}$. As such, $((z_i), n+m) \in L_2{}'$ as desired.

To show associativity, we note that both $\oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i$ and $\mathbb{Z}$ under addition are associative.

To show the existence of an identity element, let $l_0, l_1 \in L_2{}'$ with $l_0 = \big((0_i), 0\big),\ l_1 = \big((x_i), n\big)$. We claim that l_0 is an identity element and therefore must show that $l_1 l_0 = l_0 l_1 = l_1$. Observe the following.

$$\begin{aligned} l_1 l_0 &= \big((0_i), 0\big)\big((x_i), n\big) \\ &= \big((w_i), 0+n\big) \text{ where } \forall i,\ w_i = 0_i + x_{i-0} = x_i \\ &= \big((x_i), n\big) = l_1 \end{aligned}$$

$$\begin{aligned} l_0 l_1 &= \big((x_i), n\big)\big((0_i), 0\big) \\ &= \big((z_i), n+0\big) \text{ where } \forall i,\ z_i = x_i + 0_{i-n} = x_i \\ &= \big((x_i), n\big) = l_1 \end{aligned}$$

Thus, l_0 serves as an identity element in $L_2{}'$, and as such, we shall refer to l_0 as e.

To show the existence of inverses, let $l, l' \in L_2{}'$ with $l = \big((x_i), n\big)$, $l' = \big((x_{i+n}, -n)\big)$. We claim that l, l' are inverses and therefore must show that $ll' = l'l = e$. Observe the following.

$$\begin{aligned} ll' &= \big((x_{i+n}), -n\big)\big((x_i), n\big) \\ &= \big((w_i), -n+n\big) \text{ where } \forall i,\ w_i = x_{i+n} + x_{i-(-n)} = x_{i+n} + x_{i+n} = 0_i \\ &= \big((0_i), 0\big) = e \end{aligned}$$

$$\begin{aligned} l'l &= \big((x_i), n\big)\big((x_{i+n}), -n\big) \\ &= \big((z_i), n-n\big) \text{ where } \forall i,\ z_i = x_i + x_{(i+n)-n} = x_i + x_i = 0_i \\ &= \big((0_i), 0\big) = e \end{aligned}$$

Thus, l' serves as an inverse for $l \in L_2{}'$, and as such, we shall refer to l' as l^{-1}. □

That this group $L_2{}'$ is in some way linked to L_2 should be immediately apparent, as any finite subset of the integers (as in L_2) can be represented as an infinite binary string in which finitely many entries are valued 1 (as in $L_2{}'$). We prove below that these two groups are, in fact, the same up to isomorphism.

Theorem 2.1.19. $L_2{}' \cong L_2$

Proof. Define ϕ, a function from L_2 to L_2' such that, for $(S, z) \in L_2$,

$$\phi\big((S,z)\big) = \Big(\big((x_i), z\big)\Big) \quad \text{where} \quad v_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{else} \end{cases}$$

We claim that ϕ is an isomorphism. We must show that ϕ is a homomorphism, surjective, and injective.

To show that ϕ is a homomorphism, let $(S, y),\ (T, z) \in L_2$. We wish to show $\phi\big((S,y)\big)\phi\big((T,z)\big) = \phi\big((S,y)(T,z)\big)$.

$$\begin{aligned} \phi\big((S,y)(T,z)\big) &= \phi(S \cup T' - S \cap T', y+z) \\ &\qquad \text{where} \quad T' = \{t + y | t \in T\} \\ &= \big((v_i), y+z\big) \\ &\qquad \text{where} \quad v_i = 1 \text{ if } i \in S \cup T' - S \cap T',\ 0 \text{ otherwise.} \end{aligned}$$

$$\begin{aligned} \phi\big((S,y)\big)\phi\big((T,z)\big) &= \Big(\big((s_i), y\big)\Big)\Big(\big((t_i), z\big)\Big) \\ &\qquad \text{where} \quad s_i = 1 \text{ if } i \in S,\ 0 \text{ otherwise} \\ &\qquad\qquad\quad\ t_i = 1 \text{ if } i \in T,\ 0 \text{ otherwise.} \\ &= \big((u_i), y+z\big) \\ &\qquad \text{where} \quad u_i = s_i + t_{i-y} \end{aligned}$$

We claim that $u_i = v_i$.

- Note that $u_i = s_i + t_{i-y}$.
- Then $u_i = 1$ if one of the following cases are true:
  (a) $s_i = 1$ and $t_{i-y} = 0$
    - Assuming this case, we have $i \in S,\ i - y \notin T$, giving $i \notin T'$.
    - Thus, $i \in S \cup T' - S \cap T'$.

  (b) $t_{i-y} = 1$ and $s_i = 0$.
    - Assuming this case, we have $i \notin S,\ i - y \in T$, giving $i \in T'$.
    - Thus, $i \in S \cup T' - S \cap T'$.
- Thus, $u_1 = 1$ if $i \in S \cup T' - S \cap T'$, giving that $u_i = v_i$.

As $u_i = v_i$, clearly $\phi\big((S, y)\big)\phi\big((T, z)\big) = \phi\big((S, y)(T, z)\big)$, giving that ϕ is a homomorphism.

To show surjectivity, let $l \in L_2{}'$. We must show that $\exists\ (S, x) \in L_2$ such that $\phi((S, z)) = l$. We know $l = \big((x_i), z\big)$ for some $z \in \mathbb{Z}$ and some $(x_i) \in \oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i$ such that only finitely many values $x_i = 1$. As such a finite set of values exists, we let S contain these values. Then $\phi\big((S, z)\big) = l$ as desired.

To show injectivity, let $l_1 = (S, x)$, $l_2 = (T, y) \in L_2{}'$, $l_1 \neq l_2$. Then $\phi(l_1) = \big((s_i), x\big)$ where $s_i = 1$ if $i \in S$ and 0 otherwise and $\phi(l_2) = \big((t_i), y\big)$ where $t_i = 1$ if $i \in T$ and 0 otherwise. We wish to show $\phi(l_1) \neq \phi(l_2)$.

- As $l_1 \neq l_2$, either $S \neq T$ or $x \neq y$.
- If $x \neq y$, then this claim is clearly true.
- If $S \neq T$, there must exist some $j \in \mathbb{Z}$ such that $j \in S \cup T - S \cap T$. Then $s_j \neq t_j$, giving $\phi(l_1) \neq \phi(l_2)$.

□

In this section, we have discussed the dynamical system that exists around the lampstand and introduced two groups whose inner workings model the dynamics of this system. These two groups are not alone in this regard, as we will see that there are other groups with similar qualities later in this Chapter.

2.2 Presentation

The goal of this section is to prove the following theorem regarding the presentation of L_2.

Theorem 2.2.1. [6] $L_2 = \langle a, t \mid a^2, [t^j a t^{-j}, t^k a t_{-k}]\ \forall j, k \in \mathbb{Z}\rangle$.

Before we begin our discussion of the proof of this theorem, we consider the following preliminary definitions.

Definition 2.2.2. Let F_2 be a free group of rank 2 with basis $\{a, t\}$.

Note that, as F_2 is a free group, we may write $w \in F_2$ as a freely reduced word in $\{a, t\}$. As such, we know w may not contain sub-words aa^{-1}, $a^{-1}a$, tt^{-1}, $t^{-1}t$, so we may write $w = t^{l_n} a^{k_n} \cdots t^{l_2} a^{k_2} t^{l_1} a^{k_1}$ for all $w \in F_2$ where $l_i, k_j \in \mathbb{Z}$ where $1 \leq i \leq n$, $1 \leq j \leq n-1$.

Definition 2.2.3. Let N be the smallest normal subgroup of F_2 containing a^2 and $[t^{-j} a t^j, t^{-k} a t^k]\ \forall j, k \in \mathbb{Z}$.

This is to say that if M is a normal subgroup of F_2 containing a^2 and $[t^{-j} a t^j, t^{-k} a t^k]\ \forall j, k \in \mathbb{Z}$, then $N \subseteq M$, as N is the smallest such normal subgroup of F_2. Lastly, we define ψ, a map from F_2 to L_2.

Definition 2.2.4. Let $\psi : F_2 \to L_2$ be a map such that, for $w = t^{l_n} a^{k_n} \cdots t^1 a^1 \in F_2$, $\psi(w) = t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1}$, the corresponding product in L_2 where by t^{l_i}, we mean $|l_i|$ copies of t if l_i is positive and $|l_i|$ copies of t^{-1} if l_i is negative and by a^{k_i}, we mean $|k_i|$ copies of a. Recall that $t = (\emptyset, 1)$ and $a = (\{0\}, 0)$.

Having completed these preliminary definitions, we begin our proof of Theorem 2.2.1 by providing an equivalent Theorem below which makes use of the three definitions above.

Theorem 2.2.5. $L_2 \cong F_2/N$.

To prove this Theorem, we will employ the first of the Homomorphism Theorems. That is, if ψ is a homomorphism from F_2 onto L_2, then $L_2 \cong F_2/\text{Ker}(\psi)$. Thus, the following is equivalent to Theorem 2.2.5:

(1) ψ is a homomorphism.

(2) ψ is onto L_2.

(3) $\text{Ker}(\psi) = N$.

We will prove each of these statements below.

Theorem 2.2.5, Statement (1). ψ is a homomorphism.

Proof. Let $w, u \in F_2$. We must show that $\psi(wu) = \psi(w)\psi(u)$.

As $w, u \in F_2$, we may may write

$$w = t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1} \qquad u = t^{i_m} a^{j_m} \cdots t^{i_1} a^{j_1}$$

Then $\psi(w) = t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1}$ when considered as a product in L_2 and $\psi(u) = t^{i_m} a^{j_m} \cdots t^{i_1} a^{j_1}$ when considered as a product in L_2, giving that

$$\psi(w)\psi(u) = (t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1})(t^{i_m} a^{j_m} \cdots t^{i_1} a^{j_1})$$

when considered as a product in L_2. However, about wu, we may write

$$wu = t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1} t^{i_m} a^{j_m} \cdots t^{i_1} a^{j_1}$$

and this gives

$$\begin{aligned} \psi(wu) &= t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1} t^{i_m} a^{j_m} \cdots t^{i_1} a^{j_1} \\ &= (t^{l_n} a^{k_n} \cdots t^{l_1} a^{k_1})(t^{i_m} a^{j_m} \cdots t^{i_1} a^{j_1}) = \psi(w)\psi(u) \end{aligned}$$

when considered as a product in L_2. Thus, $\psi(wu) = \psi(w)\psi(u)$ as desired. □

Having shown that ψ is a homomorphism, we continue on to the second statement regarding ψ, that it is in fact onto its range L_2.

Theorem 2.2.5, Statement (2). ψ is onto L_2.

Proof. Let $g \in L_2$. We must show that there exists $w \in F_2$ with $\psi(w) = g$.

Recall that all elements of L_2 may be written as a product of a, t, t^{-1}. Thus, we may write $g = x_n \ldots x_1$ where $\forall i,\ x_i \in \{a, t, t^{-1}\}$. Consider $x_n \ldots x_1$ as a word and let $w = y_n \ldots y_1$ be the reduced form of this word. Then w is a freely reduced word in $\{a, t, t^{-1}\}$ such that, when considered as a product, $w = g$. Then, by the definition of ψ, $\psi(w) = g$ as desired. □

Thus, as required by the the First Homomorphism Theorem, ψ is a surjective homomorphism, and as such, $L_2 \cong F_2/\mathrm{Ker}(\psi)$. As such, to prove Theorem 2.2.5, it remains to show Statement (3) below.

Theorem 2.2.5, Statement (3). $\mathrm{Ker}(\psi) = N$.

Proof. We must show that $N \subseteq \mathrm{Ker}(\psi)$ and $\mathrm{Ker}(\psi) \subseteq N$.

First, we will show that $N \subseteq \mathrm{Ker}(\psi)$. Note that by Lemma 1.2.4, $N \lhd \mathrm{Ker}(\psi)$. In addition, $\psi(a^2) = e \in L_2$ and $\forall j, k \in \mathbb{Z}$, $\psi([t^{-j}at^j, t^{-k}at^k]) = e \in L_2$. Thus, $a^2, [t^{-j}at^j, t^{-k}at^k] \in \mathrm{Ker}(\psi)\ \forall j, k$. Thus, we have that $\mathrm{Ker}(\psi)$ is a normal subgroup of F_2 containing $a^2, [t^{-j}at^j, t^{-k}at^k]$. But as previously noted, N is the smallest such subgroup of F_2, and therefore, $N \subseteq \mathrm{Ker}(\psi)$ as desired.

To show $\mathrm{Ker}(\psi) \subseteq N$, we claim that $\forall w \in \mathrm{Ker}(\psi)$, w may be expressed as $\alpha_i \dots \alpha_1$ where for all i, $\alpha_i = y^{-c}\beta y^k$ where $c \in \{a, a^{-1}, t, t^{-1}\}$ and β is a product of a^2 and $[t^{-j}at^j, t^{-k}at^k]$ for some $j, k \in \mathbb{Z}$. We will not prove this result here, but rather provide several statements that motivate this claim.

Given a word $w \in \mathrm{Ker}(\psi)$, the element corresponding to w in L_2 has the lamplighter standing at index 0. This is to say that for every move the lamplighter makes away from index 0, he must make a subsequent move back toward index 0. Such an observation motivates the following claim.

Claim 2.2.6. Let $w \in \mathrm{Ker}(\psi)$. Then the exponent sum of t in w is 0.

Proof. As $w \in \mathrm{Ker}(\psi)$ and $\mathrm{Ker}(\psi) \subseteq F_2$, we may write $w = t^{l_n}a^{k_n} \cdots t^{l_1}a^{k_1}$. This gives $\psi(w) = t^{l_n}a^{k_n} \cdots t^{l_1}a^{k_1} = (S, z)$ for some $S \subseteq \mathbb{Z}$, $z \in \mathbb{Z}$. But as $w \in \mathrm{Ker}(\psi)$, $z = 0$.

We know, by the definition of operation in L_2, that every occurrence of t in w will increment z and every instance of t^{-1} in w will decrement z, so there must be an equal number of these instances. As such, if there are k occurrences of t and t^{-1}, then the exponent sum of t in w is $k - k = 0$ as desired. □

Furthermore, given $w \in \mathrm{Ker}(\psi)$, the element corresponding to w in L_2 has no lamps illuminated. This is to say that if in w the lamplighter were to turn some lamp on, it must at some point subsequently turn it off. This observation motivates the following claim.

Claim 2.2.7. Let $w = y_\alpha \cdots y_1 \in \mathrm{Ker}(\psi)$ where $y_i \in \{a, a^{-1}, t, t^{-1}\}$ $\forall i$. Then $\forall p$ such that $\psi(y_p \cdots y_1) = (\{l\}, z)$ for some $l, z \in \mathbb{Z}$, there exists $q > p$ such that $y_{q+1} \in \{a, a^{-1}\}$ and $\psi(y_q \cdots y_1) = (T, l)$ for some $T \subseteq \mathbb{Z}$.

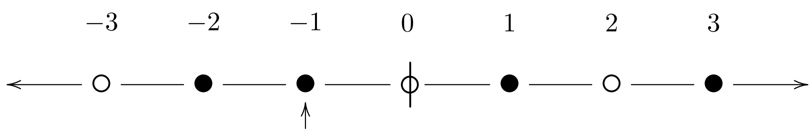
Proof. Suppose to the contrary that, given some $w = y_\alpha \cdots y_1 \in \mathrm{Ker}(\psi)$, there exists p such that $\psi(y_p \cdots y_1) = (\{l\}, z)$ without the existence of such a q as described above. Then for all r with $p \le r \le \alpha$, $\psi(y_r \cdots y_1) = (U, z')$ with $l \in U$. Thus, if $\psi(w) = \psi(y_\alpha \cdots y_1) = (S, z'')$, then $l \in S$. But as $w \in \mathrm{Ker}(\psi)$, $\psi(w) = \{\emptyset, 0\}$, a contradiction. □

To show the injectivity of this map is a massively complicated undertaking that we will not discuss here. Rather, we will discuss a more graphically intuitive topic, namely the word length of elements in the Lamplighter Group. This discussion begins in the following section.

2.3 Word Length in L_2

In this section, we present a formula for the word length of any element in L_2. That is, for $l \in L_2$, this formula will find the length of the shortest word w in $\{a, t, t^{-1}\}$ such that $l = w$ when w is considered as a product. Furthermore, in addition to specifying the length of such a word, discussion in this section will yield the word itself.

We begin with an example. Let $g_0 = (\{-2, -1, 1, 3\}, -1) \in L_2$, shown below.



By the alternate proof of Theorem 2.1.11, we have seen that we may express this element as a word in $\{a, t, t^{-1}\}$ by multiplying our generators by the identity as follows. For each index at which a bulb is illuminated in g_0

- move the lamplighter to that index,
- change the state of the bulb at that index,
- return the lamplighter to the origin,

and then finally move the lamplighter to the index of his final destination. We may express this process as the following word.

$$t^{-1} \left(t^1 a t^{-1}\right) \left(t^{-3} a t^3\right) \left(t^2 a t^{-2}\right) \left(t^{-1} a t^1\right)$$

While this word has length 19, we may reduce it by combining the exponents of adjacent generators and obtain the following word, in which the lamplighter does not return to the origin in between stops at other indices.

$$a\ t^{-4}\ a\ t^5\ a\ t^{-3}\ a\ t$$

This word corresponds to stopping the Lamplighter at indices 1, -2, 3, and -1 (in that order) to illuminate each bulb, and then finally stopping him at his final destination -1. Summing the exponents of the reduced form of this word, we see that this word has a length of 17.

We have seen, however, that there are many ways to construct a word to obtain any individual element of L_2. In the following discussion, we present, for any $l \in L_2$, two words in $\{a, t, t^{-1}\}$ that when considered as a product, equal l. We will prove that one of these words has minimal length. Before we do so, however, we provide a more intricate description of the elements in L_2 that we will use to construct words with minimal lengths.

Definition 2.3.1. Let $g = (S, z) \in L_2$. Then we define $S^+, S^- \subseteq S$ as follows:

$$S^+ = \{s \in S | s \geq 0\} \qquad S^- = \{s \in S | s < 0\}$$

Letting $k = |S^+|$ and $j = |S^-|$, we will label the indices in these subsets as

$$\begin{aligned} S^+ &= \{i_1, \ldots, i_k\} && \text{where } i_1 \leq \cdots \leq i_k \\ S^- &= \{-j_1, \ldots, -j_l\} && \text{where } -j_1 \geq \cdots \geq -j_l \end{aligned}$$

Note that if $k = 0$, then $S^+ = \emptyset$, meaning that there will be no bulbs of non-negative index in g, and thus that i_k does not exist, since i_k is the illuminated bulb with the greatest non-negative index. Similarly, if $j = 0$, then $S^- = \emptyset$, giving no illuminated bulbs of negative index in g, and thus j_l does not exist. Returning to our example, $g_0 = (\{-2, -1, 1, 3\}, -1)$, we note that $S^+ = \{1, 3\}$, $S^- = \{-1, -2\}$, giving $i_1 = 1$, $i_2 = 3$, $j_1 = 1$, $j_2 = 2$.

Next, we present the first of our two candidates for the minimal length word by continuing with our example g_0. Note the following word.

$$t^{-1} \left(t^2 a t^{-2}\right) \left(t^1 a t^{-1}\right) \left(t^{-3} a t^3\right) \left(t^{-1} a t^1\right)$$

This word, when considered as a product, is equivalent to g_0. Reducing this word, we obtain

$$t \ a \ t \ a \ t^{-4} \ a \ t^2 \ a \ t$$

This word corresponds to first stopping the Lamplighter at each desired non-negative index (in increasing order) to turn its bulb on, then doing the same at each desired negative index (in decreasing order), and then finally stopping the Lamplighter at his final destination. Summing the exponents of the reduced form of this word above, we see that this word has a length of 13. However, such a process can be described in general for any element of the Lamplighter Group.

Given an element $g = (S, z) \in L_2$ where $S^+ = \{i_1, \ldots, i_k\}$, $S^- = \{-j_1, \ldots, -j_l\}$, we may describe the process above in general as follows:

- Stop at bulbs at indices $i_1 \ldots, i_k$, turning on each.
- Stop at bulbs at indices $-j_1, \ldots, j_l$, turning on each.
- Stop at the final destination bulb at index z.

This process corresponds to a word whose construction is outlined in following definition.

Definition 2.3.2. Let $g = (S, z) \in L_2$ where $S^+ = \{i_1, \ldots, i_k\}$, $S^- = \{-j_1, \ldots, -j_l\}$. Then the *right-left word* of g, denoted $p^+(g)$, is the reduced form of the following word:

$$t^z \left[(t^{j_l} a t^{-j_l}) \cdots (t^{j_1} a t^{-j_1})\right] \left[(t^{-i_k} a t^{i_k}) \cdots (t^{-i_1} a t^{i_1})\right]$$

Reduced, this gives us

$$p^+(g) = t^{j_l+z}at^{-j_l+j_{l-1}}a \cdots at^{-j_2+j_1}at^{-i_k-j_1}at^{i_k-i_{k-1}}a \cdots at^{i_2-i_1}at^{i_1}$$

Note that for our example, we have already computed $l(p^+(g_0)) = 13$. We next go on to compute the length of $p^+(g)$ for any $g \in L_2$.

Claim 2.3.3. Let $g \in L_2$. Then $l(p^+(g)) = k + l + 2i_k + j_l + |z + j_l|$.

Proof. To find the length of $p^+(g)$, we simply count the generators separately. There are $k+l$ occurrences of a. Furthermore, note that the sum of the absolute values of the exponents of t can be written as follows:

$$\begin{aligned} i_1+\sum_{x=2}^{k}(i_x - i_{x-1}) \ + i_k + j_1 + \sum_{x=2}^{l}(j_x - j_{x-1}) + |j_l + z| \\ = i_1 + (i_k - i_1) + i_k + j_1 + (j_l - j_1) + |j_l + z| \\ = 2i_k + j_l + |j_l + z|. \end{aligned}$$

Thus, we have

$$l(p^+(g)) = k + l + 2i_k + j_l + |z + j_l|.$$

$\square$

The right-left word corresponds to sending the lamplighter to illuminate positive bulbs in ascending order before illuminating negative bulbs in descending order. We could, however, construct a word that directs the lamplighter in the reverse direction, first illuminating negative bulbs in descending order and then illuminating positive bulbs in ascending order. Implementing such a strategy with our example element g_0 would yield the following word.

$$t^{-1}\left(t^{-3}at^3\right)\left(t^{-1}at^1\right)\left(t^2at^{-2}\right)\left(t^1at^{-1}\right)$$

Reducing this word, we obtain

$$t^{-4}\ a\ t^2\ a\ t^3\ a\ t^{-1}\ a\ t^{-1}$$

We may count the generators of this reduced word to attain its word length of 15. Again, we may use this process to construct a corresponding word for any element of the Lamplighter Group. That word, which is in fact our second candidate for the minimal length word, is outlined in the following definition.

Definition 2.3.4. Let $g = (S, z) \in L_2$ where $S^+ = \{i_1, \ldots, i_k\}$, $S^- = \{-j_1, \ldots, -j_l\}$. Then the *left-right word* of g, denoted $p^-(g)$, is the reduced form of the following word:

$$t^z\left[(t^{-i_k}at^{i_k}) \cdots (t^{-i_1}at^{i_1})\right]\left[(t^{j_l}at^{-j_l}) \cdots (t^{j_1}at^{-j_1})\right]$$

Reducing this word, we obtain the left-right word.

$$p^-(g) = t^{-i_k+z}at^{i_k-i_{k-1}}a \cdots at^{i_2-i_1}t^{i_1+j_l}at^{-j_l+j_{l-1}}a \cdots at^{-j_2+j_1}at^{-j_1}$$

Again, we have already computed $l(p^-(g_0)) = 15$, and through a proof identical to that of Claim 2.3.3 regarding length of the right-left word, we may obtain the following formula for length of the left-right word.

Claim 2.3.5. $l(p^-(g)) = k + l + 2j_l + i_k + |z - i_k|$.

We have $l(p^+(g_0)) = 13 \geq 15 = l(p^-(g_0))$. Given some element in L_2, however, it will not necessarily be the case that the length of its right-left word will be greater than the length of its left-right word. Consider $g_1 = (\{-2, -1, 1, 3\}, 1) \in L_2$. Given S, the vertex set of g_1, $S^+ = \{1, 3\}$, $S^- = \{-1, -2\}$, giving $k = l = 2$, $i_k = 3$, $j_l = 2$, and $z = 1$. We compute the lengths of the right-left word and the left right word of g_1 as follows:

$$\begin{aligned} l(p^+(g_1)) &= k + l + 2i_k + j_l + |z + j_l| \\ &= 2 + 2 + 2(3) + 2 + |1 + 2| \\ &= 15 \\ l(p^-(g_1)) &= k + l + 2j_l + i_k + |z - i_k| \\ &= 2 + 2 + 2(2) + 3 + |1 - 3| \\ &= 13 \end{aligned}$$

Thus, we have $l(p^+(g_1)) = 15 \leq 13 = l(p^-(g_1))$, and as such, there exist elements for which either word may be of lesser length. However, we note that the only difference between these two example elements is that the lamplighter stands at a negative index in g_0 and at a positive index in g_1. This observation motivates the following claim.

Theorem 2.3.6. Let $g = (S, z) \in L_2$ where $S^+ = \{i_1, \ldots, i_k\}$, $S^- = \{-j_1, \ldots, -j_l\}$. Then

(i) if $z \geq 0$, $l(p^-(g)) \leq l(p^+(g))$

(ii) if $z \leq 0$, $l(p^+(g)) \leq l(p^-(g))$

Proof. Let $g \in L_2$ as above. We will consider the cases (i) and (ii) as above.

Case (i). Assume $z \geq 0$. Then

$$\begin{aligned} l(p^+(g)) &= k + l + 2i_k + j_l + |z + j_l| \\ &= k + 1 + 2i_k + 2j_l + z \end{aligned}$$

We wish to show that $l(p^-(g)) \leq l(p^+(g))$. We consider three sub cases:

(a) $z = j_l$
Then

$$\begin{aligned} l(p^-(g)) &= k + l + 2j_l + i_k + |z - i_k| \\ &= k + l + i_k + 2j_l \\ &\leq k + l + 2i_k + 2j_l + z = l(p^+(g)). \end{aligned}$$

(b) $z < i_k$
Then

$$\begin{aligned} l(p^-(g)) &= k + l + 2j_l + i_k + |z - i_k| \\ &= k + l + 2j_l + 2i_k - z \\ &\leq k + l + 2i_k + 2j_l + z = l(p^+(g)). \end{aligned}$$

(c) $z > i_k$
Then

$$\begin{aligned} l(p^-(g)) &= k + l + 2j_l + i_k + |z - i_k| \\ &= k + l + 2j_l + z \\ &\leq k + l + 2i_k + 2j_l + z = l(p^+(g)). \end{aligned}$$

Therefore, for $z \geq 0$, $l(p^+(g)) \geq l(p^-(g))$, as desired.

Case (ii).
Assume $z \leq 0$. Then $\exists z' \geq 0$ such that $-z = z'$. Thus

$$\begin{aligned} l(p^-(g)) &= k + l + 2j_l + i_k + |-z' - i_k| \\ &= k + l + 2i_k + 2j_l + z' \end{aligned}$$

We wish to show that $l(p^+(g)) \leq l(p^-(g))$. Again, we consider three sub-cases:

(a) $z' = j_l$.
Then

$$\begin{aligned} l(p+(g)) &= k + l + 2i_k + j_l + |j_l - z'| \\ &= k + l + 2i_k + j_l \\ &\leq k + l + 2i_k + 2j_l + z' = l(p^-(g)). \end{aligned}$$

(b) $z' < j_l$
Then

$$\begin{aligned} l(p+(g)) &= k + l + 2i_k + j_l + |j_l - z'| \\ &= k + l + 2i_k + 2j_l - z' \\ &\leq k + l + 2i_k + 2j_l + z' = l(p^-(g)). \end{aligned}$$

(c) $z' > j_l$
Then

$$\begin{aligned} l(p+(g)) &= k + l + 2i_k + j_l + |j_l - z'| \\ &= k + l + 2i_k + z' \\ &\leq k + l + 2i_k + 2j_l + z' = l(p^-(g)). \end{aligned}$$

Therefore, when $z \leq 0$, $l(p^-(g)) \geq l(p+(g))$, which proves our claim. $\square$

As a corollary, note that if $z = 0$, then $l(p^+(g)) = l(p^-(g))$.

Thus, for any $g = (S, z) \in L_2$, we have presented two candidates for the minimal length word w such that $w = g$, namely the right-left word and the left-right word. We have also shown that if $z > 0$, then the left-right word of g is shorter than its right-left word and that if $z < 0$, then the right-left word of g is shorter than its left-right word. Consider the following definition.

Definition 2.3.7. Let D be a function from the Lamplighter group to $\mathbb{Z}^+$ such that for all $g \in L_2$, $D(g) = \min\{l(p^+(g)), l(p^-(g))\}$.

As a result of Theorem 2.3.6 we have that if $z \geq 0$, then $l(p^+(g)) \leq l(p^-(g))$ and therefore $D(g) = l(p^-(g))$. Similarly, we have that if $z \leq 0$, then $l(p^-(g)) \leq l(p^+(g))$ and therefore $D(g) = l(p^+(g))$. We conclude this section with a proof that $D(g)$ is in fact a formula for word length in the Lamplighter Group.

Theorem 2.3.8. (Word Length Formula) Let $g \in L_2$. Then $D(g) = l(g)$.

Proof. Let $g \in L_2$ where $S^+ = \{i_1, \ldots, i_k\}$, $S^- = \{-j_1, \ldots, -j_l\}$. We know that $D(g)$ is the length of a word of a, t (specifically, either $p^+(g)$ or $p^-(g)$). But by definition, $l(g)$ is the length of the shortest word of a, t, and therefore $l(g) \leq D(g)$. Thus, it remains to show $l(g) \geq D(g)$.

Let $w = x_y \cdots x_1$ be be the shortest word in $\{a, t, t^{-1}\}$ such that, when considered as a product, $w = g$. We may compute $l(g) = l(w)$ by counting the generators in w, and we do so in each of the four following cases:

(a) $S^+ = \emptyset,\ S^- = \emptyset$.
This case represents that of an element g in which no bulbs are illuminated. As such, for the lamplighter to reach his final destination index z, there must be at least $|z|$ occurrences of t in w, giving $l(g) = l(w) \geq |z|$. However $p^+(g) = p^-(g) = t^z$, and therefore, $D(g) = |z|$, giving $l(g) \geq |z| = D(g)$ as desired.

(b) $S^+ \neq \emptyset,\ S^- = \emptyset$.
This case represents that of an element g in which only bulbs at non-negative indices are illuminated. There must be at least k occurrences of a in w, one for each of the illuminated bulbs, one of which being the bulb at index i_k. For the lamplighter to illuminate this bulb, there must exist some suffix w_n of w with $w_n = x_n \cdots x_1 = (S', i_k)$ where $x_n = a$. When considered as a product, w_n is an element of L_2 in which the lamplighter is standing at index i_k. For the lamplighter to reach such an index, there must be at least i_k occurrences of t in w_n.

We now consider two sub-cases:

(i) $z \geq 0$.
As $z \geq 0$, $D(g) = l(p^-(g)) = k + l + 2j_l + i_k + |z - i_k|$, but since $l = 0$ and j_l does not exist, we have $D(g) = k + i_k + |z - i_k|$. But in order for the lamplighter to reach his final destination at index z, there must be at least $|z - i_k|$ occurrences of t in the prefix $w_n{}' = x_k \cdots x_{n+1}$. This gives $l(g) \geq k + i_k + |z - i_k| = D(g)$ as desired.

(ii) $z < 0$.
As $z < 0$, $D(g) = l(p^+(g)) = k + l + 2i_k + j_l + |z + j_l| = k + 2i_k + |z|$ since, again $l = 0$ and j_l does not exist. But in order for the lamplighter to reach his final destination at index z, there must be at least $i_k + |z|$ occurrences of t in the prefix $w_n{}'$, giving $l(g) \geq k + 2i_k + |z| = D(g)$ as desired again.

(c) $S^+ = \emptyset,\ S^- \neq \emptyset$.
This case will require an argument similar to that of the previous case and represents that of an element g in which only bulbs at negative indices are illuminated. There must be at least l occurrences of a in w, one for each illuminated bulb, including the bulb at index $-j_l$. In order to illuminate this bulb, there must exist some suffix $w_m = x_m \cdots x_1 = (S'', -j_l)$ where $x_m = a$. For the lamplighter to reach index $-j_l$, there must be at least j_l occurrences of t in w_m.

Again, we consider two sub-cases:

(i) $z \geq 0$
As $z \geq 0$, $D(g) = l(p^-(g)) = k + l + 2j_l + i_k + |z - i_k|$, but since $k = 0$ and i_k does not exist, we have $D(g) = l + 2j_l + |z|$. But in order for the lamplighter to reach his final destination at index z, there must be at least $i_k + |z|$ occurrences of t in the prefix $w_m{}' = x_1 \cdots x_{m+1}$, giving $l(g) \geq l + 2j_l + |z|$ as desired.

(ii) $z < 0$
As $z < 0$, $D(g) = l(p^+(g)) = k + l + 2i_k + j_l + |z + j_l| = l + j_l + |z + j_l|$. But in order for the lamplighter to reach his final destination index z, there must be at least $|z + j_l|$ occurrences of t in the prefix $w_m{}'$, giving $l(g) \geq l + j_l + |z + j_l| = D(g)$ as desired once more.

(d) $S^+ \neq \emptyset,\ S^- \neq \emptyset$.
This case represents an element in which both negative and non-negative bulbs are illuminated. Here, there must be at least $k + l$ occurrences of a in w, again one for each bulb, including the bulbs at indices i_k and $-j_l$. To illuminate these specific bulbs, there must exist n, m such that, when considered as a product

$$x_n \cdots x_1 = (S', i_k) \quad \text{and} \quad x_m \cdots x_1 = (S'', -j_l)$$

where $x_n = x_m = a$. Let n, m be the smallest such values for which this is true and denote $w_n = x_n \cdots x_1$ and $w_m = x_m \cdots x_1$.

Note that $i_k \neq j_l$ since $S^+ \cap S^- = \emptyset$, and as such, either $n > m$ or $n < m$. We explore these two sub-cases here:

(i) $n < m$.
In this case, $w = w_0 w_n' w_n$ where $w_n' w_n = w_m$ and w_0 is a prefix of w. In order for the lamplighter to reach index i_k, there must be at least i_k occurrences of t in w_n. From there, for the lamplighter to reach index $-j_l$, there must be at least $i_k + j_l$ occurrences of t in w_n'. Again, we consider the following cases regarding the index of lamplighter's final destination.

  i. $z \geq 0$.
  As $z \geq 0$, $D(g) = l(p^-(g)) = k+l+2j_l+i_k+|z-i_k| = k+l+2j_l+z$ since $z, i_k \geq 0$. For the lamplighter to reach index z, there must be at least an additional $j_l + |z| = j_l + z$ occurrences of t in w_0, giving $l(g) \geq k + l + 2i_k + 2j_l + z \geq k + l + 2j_l + z = D(g)$ as desired.

  ii. $z < 0$.
  As $z < 0$, $D(g) = l(p^+(g)) = k + l + 2i_k + j_l + |z + j_l|$. But for the lamplighter to reach index z, there must be an additional $|z + j_l|$ occurrences of t in w_0, giving $l(g) \geq k+l+2i_k+j_l+|z+j_l| = D(g)$ as desired.

(ii) $n > m$.
In this case $w = w_0 w_m' w_m$ where $w_m' w_m = w_n$ and w_0 is again a prefix of w. We count at least j_l occurrences of t in w_m in order for the lamplighter to reach index $-j_l$ and an additional $j_l + i_k$ occurrences of t in w_m' in order for the lamplighter to then reach index i_k. Once, we consider cases regarding z:

  i. $z \geq 0$.
  As $z \geq 0$, $D(g) = l(p^-(g)) = k + l + 2j_l + i_k + |z - i_k|$, and for the lamplighter to reach index z from index i_k, there must be an additional $|z - i_k|$ occurrences of t in w_0, giving $l(g) \geq k + l + 2j_l + i_k + |z - i_k| = D(g)$ as desired.

  ii. $z < 0$.
  As $z < 0$, $D(g) = l(p^+(g)) = k + l + 2i_k + j_l + |z + j_l| \geq k + l + 2i_k + 2j_l + |z|$. For the lamplighter to reach index z from i_k, there must be an additional $i_k + |z|$ occurrences of t in w_0, giving $l(g) \geq k + l + 2j_l + 2i_k + |z| \geq D(g)$ as desired once more.

□

Thus, we have shown the word length formula for elements of the Lamplighter Group. We will use this formula later in showing both the existence of dead-end elements in L_2 and the depth of these elements. This discussion begins in the following section.

2.4 Dead-End Elements of Arbitrary Depth

In this section, we explore dead-end elements in the Lamplighter Group. That is, when multiplying these elements by a single generator, the length of the product is less than that of the original element.

Theorem 2.4.1. [6, Theorem 0.1] The Lamplighter Group contains dead-end elements of arbitrary depth. That is, for every $n \geq 1$, there exists some element $d_n \in L_2$ satisfying the following statements:

(1) d_n is a dead-end element

(2) d_n has depth at least n

We begin with the definition of an element $d_n \in L_2$.

Definition 2.4.2. Let $d_n = (N_n, 0)$ where $N_n = \{-n, \ldots, n\}$ for $n \in \mathbb{Z}$.

This element d_n corresponds to an element in which all bulbs between indices $-n$ and n (inclusive) are illuminated and the lamplighter standing at the origin. We compute its length below.

Note that ${N_n}^+ = \{0, \ldots, n\}$ and ${N_n}^- = \{-1, \ldots, -n\}$, giving $k = n + 1$, $l = i_k = j_l = n$. Since $z = 0$, we know that $l(p^+(d_n)) = l(p^-(d_n))$, and therefore we need only compute one of these values to compute $l(d_n) = \min\{l(p^+(d_n)), l(p^-(d_n))\}$.

$$\begin{aligned} l(d_n) = l(p^+(d_n)) &= k + l + 2i_k + j_l + |z + j_l| \\ &= (n+1) + n + 2n + n + |0 + n| \\ &= 6n + 1 \end{aligned}$$

Theorem 2.4.1, Statement (1). d_n is a dead-end element.

Proof. Let $g \in L_2$ such that $d(d_n, g) = 1$. We must show that $l(d_n) \geq l(g)$. There exist three such possible elements g in L_2 since to find an element whose distance from d_n is 1, we need only multiply g by one of our three generators in $\{a, t, t^{-1}\}$. We examine each element as a separate case:

(i) $g = ad_n = (S, 0)$ where $S = \{-n, \ldots, -1, 1, \ldots, n\}$.
Note that $S^+ = \{1, \ldots, n\}$ and $S^- = \{-1, \ldots, -n\}$, giving $k = l = i_k = j_l = n$. Again, $z = 0$, so $l(g) = l(p^+(g))$, which we compute below.

$$\begin{aligned} l(g) = l(p^+(g)) &= k + l + 2i_k + j_l + |z + j_l| \\ &= n + n + 2n + n + |0 + n| \\ &= 6n \end{aligned}$$

Thus, for such a g

$$l(d_n) = 6n + 1 \geq 6n = l(g)$$

(ii) $g = td_n = (N_n, 1)$.
Again, we have ${N_n}^+ = \{0, \dots, n\}$ and ${N_n}^- = \{-1, \dots, -n\}$, giving $k = n+1$, $l = i_k = j_l = n$. Since $z = 1 \geq 0$, $l(g) = l(p^-(g))$, which we compute below.

$$\begin{aligned} l(g) = l(p^-(g)) &= k + l + 2j_l + i_k + |z - i_k| \\ &= n + n + 2n + n + |1 - n| \\ &\leq 5n + (n-1) = 6n - 1 \end{aligned}$$

Thus, for such a g

$$l(d_n) = 6n + 1 \geq 6n - 1 \geq l(g)$$

(iii) $g = t^{-1}d_n = (N_n, -1)$.
This case differs from the previous case only in that $z = -1 \leq 0$, giving $l(g) = l(p^+(g))$, which we compute below.

$$\begin{aligned} l(g) = l(p^+(g)) &= k + l + 2i_k + j_l + |z + j_l| \\ &= n + n + 2n + n + |-1 + n| \\ &\leq 5n + (-1 + n) = 6n - 1 \end{aligned}$$

And so, for such a g

$$l(d_n) = 6n + 1 \geq 6n - 1 \geq l(g)$$

Thus, for all $g \in L_2$ with $d(d_n, g) = 1$, $l(d_n) = l(g)$, and therefore, d_n is a dead-end element in L_2. □

We have clearly proven that d_n is a dead-end element in L_2 for any n. Thus, it remains to show that d_n is of depth of at least n. To prove this result, we consider d_n as a member of the following set:

Definition 2.4.3. For $n \in \mathbb{Z}$, we define $H_n = \{(S, z) | S \subseteq N_n, z \in N_n\}$.

Clearly, $d_n \in H_n$. We will show that d_n has maximal length in H_n.

Lemma 2.4.4. For all $h \in H_n$, $l(d_n) \geq l(h)$.

Proof. Let $h = (S, z) \in H_n$. Then $S^+ \subseteq {N_n}^+ = \{0, \dots, n\}$ and $S^- \subseteq {N_n}^- = \{-1, \dots, -n\}$. As such, we may note the following about the values required to compute the length of the right-left word and left-right word of h.

$$k \leq n + 1 \qquad l, i_k, j_l \leq n \qquad -n \leq z \leq n$$

We consider the following cases for different values of z.

(i) $0 \leq z \leq n$.
Then $l(h) = l(p^-(h))$, which we compute below:

$$\begin{aligned} l(h) = l(p^-(h)) &= k + l + 2j_l + i_k + |z - i_k| \\ &\leq n + n + 2n + n + |n| = 6n \end{aligned}$$

(ii) $0 \geq z \geq -n$.
Then $l(h) = l(p^+(h))$, which we compute below:

$$\begin{aligned} l(h) = l(p^+(h)) &= k + l + 2i_k + j_l + |z + j_l| \\ &\leq n + n + 2n + n + |n| = 6n \end{aligned}$$

Thus, for such an element h in either case,

$$l(d_n) = 6n + 1 \geq 6n \geq l(h)$$

as desired. □

Thus, d_n has maximal length in H_n. To prove our Theorem, however, it remains to show that d_n has depth at least n, or in other words that for all $g \in L_2$ such that $l(g) \geq l(n)$, $d(d_n, g) \geq n$. However, according to the Lemma above, we know that $g \notin H_n$. This motivates our proof below.

Theorem 2.4.1, Statement (2). d_n has depth at least n

Proof. Let $g = (S, z) \in L_2$ such that $l(g) \geq l(d_n)$ and let $w = x_k \dots x_1$ be a word in $\{a, t, t^{-1}\}$ such that, when considered as a product, $wd_n = g$. We wish to show that $l(w) \geq n$.

As $l(g) \geq l(d_n)$, by Lemma 2.4.4, $g = (S, z) \notin H_n$, and therefore it must be the case that either

(i) $z \geq n + 1$ or $z \leq -n - 1$

(ii) $\exists s \in S$ such that $s \geq n + 1$ or $s \leq -n - 1$.

We consider these cases below.

(i) Assume $z \geq n + 1$.
Then there must exist some $i \leq k$ such that $x_i \dots x_1 d_n = (S', n + 1)$ for some $S' \subseteq \mathbb{Z}$. But as $d_n = (N_n, 0)$ with the lamplighter at the origin, there must be at least $n+1$ occurrences of t in $x_i \dots x_1$, giving $l(w) \geq n+1 \geq n$ as desired. Note that a similar argument can be made if $z \leq -n - 1$.

(ii) Assume $\exists s \in S$ such that $s \geq n + 1$.
Then there must exist some $i \leq k$ such that $x_i \dots x_1 d_n = (S', s)$ and $x_i = a$ for some $S' \subseteq \mathbb{Z}$. But as $s \geq n+1$, there must be at least $n+1$ occurrences of t in $x_i \dots x_1$ as above, giving $l(w) \geq n + 1 \geq n$ as desired again. Again, a similar argument can be made if $s \leq -n - 1$.

□

Thus, for $n \in \mathbb{Z}$, we have shown that d_n has depth at least n. While this proves that the Lamplighter Group contains dead-end elements of arbitrary depth, a stronger statement can be made regarding the depth of d_n.

Theorem 2.4.5. For $n \in \mathbb{Z}^+$, d_n has depth $2n+1$. In other words, regarding d_n, we may state the following:

(1) There exists a word $w = x_{2n+1} \cdots x_1$ where $\forall i,\ x_i \in \{a, t, t^{-1}\}$ such that $l(wd_n) > l(d_n)$.

(2) For all words $w = x_\alpha \cdots x_1$ where $\alpha \leq 2n$ and $\forall i,\ x_i \in \{a, t, t^{-1}\}$, $l(wd_n) \leq l(d_n)$.

We will prove the first of these two statements now.

Theorem 2.4.5., Statement (1). There exists a word $w = x_{2n+1} \cdots x_1$ where $\forall i,\ x_i \in \{a, t, t^{-1}\}$ such that $l(wd_n) > l(d_n)$.

Proof. Consider $w = t^{2n+1}$. We claim that $l(wd_n) > l(d_n) = 6n+1$. We may compute $wd_n = (N_n, 2n+1)$, which gives:

$$N_n^+ = \{0, \ldots, n\} \qquad\qquad N_n^- = \{-n, \ldots, -1\}$$

and therefore $k + l = 2n+1,\ i_k = j_l = n,\ z = 2n+1$. As $z \geq 0$, $l(wd_n) = l(p^-(wd_n))$, which we compute as follows.

$$\begin{aligned} l(wd_n) = l(p^-(wd_n)) &= k + l + 2j_l + i_k + |z - i_k| \\ &= 2n + 1 + 2n + n + |2n + 1 - n| \\ &= 5n + 1 + |n + 1| \\ &= 6n + 2 \end{aligned}$$

Therefore, as desired, we have

$$l(wd_n) = 6n + 2 > 6n + 1 = l(d_n).$$

□

To prove the second statement in Theorem 2.4.5, we will make use of two functions $\mathcal{I}$ and $\mathcal{O}$ as defined below.

Definition 2.4.6. Let $w = \alpha_1 \cdots \alpha_n$ be a word in some set S. Then for $s \in S$, $\mathcal{O}_s(w) = |\{i | \alpha_i = s \text{ or } \alpha_i = s^{-1}\}|$. In other words, $O_s(w)$ is the number of occurrences of the generator s and its inverse in w.

For example, let $w = at^{-4}at^5at^{-3}at$, the first word mentioned in the previous section. Then $\mathcal{O}_a(w) = 4$ and $\mathcal{O}_t(w) = 13$. Furthermore, $\mathcal{O}_a + \mathcal{O}_t = 17 = l(w)$, a result which motivates the following Lemma.

Lemma 2.4.7. Let G be a group with generating set $S = \{s_1, \ldots, s_k\}$. Then for all $w = \alpha_1 \cdots \alpha_n$ where w is a word in S, $n = \sum_{i=1}^{k} \mathcal{O}_{s_i}(w)$.

Proof. Let G be a group with generating set S as above and let $w = \alpha_1 \cdots \alpha_n$ be a word in S. Then we may partition the integers $1, \ldots, n$, into sets $S_j = \{i | \alpha_i = s_j \text{ or } \alpha_i = {s_j}^{-1}\}$. Certainly, the sum of the cardinalities of these sets will be n. Furthermore, by definition of $\mathcal{O}$, $|S_j| = \mathcal{O}_{s_j}(w)$, and therefore, as desired, we have

$$\begin{aligned} n &= |S_1| + \cdots + |S_k| \\ &= \mathcal{O}_{s_1}(w) + \cdots + \mathcal{O}_{s_k}(w) \\ &= \sum_{i=1}^{k} \mathcal{O}_{s_i}(w). \end{aligned}$$

□

This result is not surprising, but will assist in the proof of Theorem 2.4.5. We continue with the definition of $\mathcal{I}$ below.

Definition 2.4.8. Let $\mathcal{I}$ be a function from L_2 to L_2 such that, for $g = (S, z) \in L_2$ with $S^+ = \{i_1, \ldots, i_k\}$ and $S^- = \{-j_1, \ldots, -j_l\}$, we define $\mathcal{I}(g) = (S', z')$ where $S' = \{-s | s \in S\}$ and $z' = -z$.

Given the lampstand configuration corresponding to some element g, we may flip this configuration about its origin to produce the configuration corresponding to $\mathcal{I}(g)$. That g and $\mathcal{I}(g)$ are of equal length is not surprising, and such a result is proven below.

Lemma 2.4.9. $l(g) = l(\mathcal{I}(g))$.

Proof. Let $g \in L_2$. Then $\mathcal{I}(g) = (S', z')$ where $S' = \{-s | s \in S\}$, giving $(S')^+ = \{i_{1'}, \ldots, i_{k'}\}$, $(S')^- = \{-j_{1'}, \ldots, -j_{l'}\}$. Note that

$$k' = l,\ l' = k,\ i_{k'} = j_l,\ j_{l'} = i_k,\ z' = -z$$

We have $l(\mathcal{I}(g)) = \min\{l(p^+(\mathcal{I}(g))), l(p^-(\mathcal{I}(g)))\}$. We compute the length of these words below.

$$\begin{aligned} l(p^+(\mathcal{I}(g))) &= k' + l' + 2i_{k'} + j_{l'} + |z + j_{l'}| \\ &= l + k + 2j_l + i_k + |-z + i_k| \\ &= k + l + 2j_l + i_k + |z - i_k| \\ &= l(p^-(g)) \end{aligned}$$

$$\begin{aligned} l(p^-(\mathcal{I}(g))) &= k' + l' + 2j_{l'} + i_{k'} + |z - i_{k'}| \\ &= l + k + 2i_k + j_l + |-z - j_l| \\ &= k + l + 2i_k + j_l + |z + j_l| \\ &= l(p^+(g)) \end{aligned}$$

Therefore, we may say the following regarding $l(\mathcal{I}(g))$.

$$\begin{aligned} l(\mathcal{I}(g)) &= \min\{l(p^+(\mathcal{I}(g))), l(p^-(\mathcal{I}(g)))\} \\ &= \min\{l(p^-(g)), l(p^+(g))\} \\ &= l(g) \end{aligned}$$

□

Having defined both $\mathcal{O}$ and $\mathcal{I}$, we continue our proof of Theorem 2.4.5 in which we will use both of these newly defined functions and the Lemmas that they motivated.

Theorem 2.4.5., Statement (2). For all words $w = x_\alpha \cdots x_1$ where $\alpha \leq 2n$ and $\forall i,\ x_i \in \{a, t, t^{-1}\}$, $l(wd_n) \leq l(d_n)$.

Proof. Let $w = x_\alpha \cdots x_1$ with $\alpha \leq 2n$ and such that $\forall i,\ x_i \in \{a, t, t^{-1}\}$. If, when considered as a product, $wd_n = (S, z) \in H_n$, then as d_n has maximal length in H_n, $l(wd_n) \leq l(d_n)$. Thus, we will assume $wd_n \notin H_n$.

Furthermore, we will assume without loss of generality that $z \geq 0$, since if $z < 0$, we may consider $\mathcal{I}(wd_n)$, an element whose length is equal to that of wd_n and in which the lamplighter stands at index $-z > 0$, reducing the case to that of our assumption above. As $z \geq 0$, $l(wd_n) = l(p^-(wd_n))$. Thus, we must show the following:

$$l(wd_n) = l(p^-(wd_n)) = k + l + 2j_l + i_k + |z - i_k| \leq l(d_n) = 6n + 1$$

To show this result, we will first prove the following claims:

(i) $z \leq 2n$

(ii) $j_l = n$

(iii) $i_k + |z - i_k| \leq 2n$

Claim 2.4.10. $z \leq 2n$.

Proof. Suppose to the contrary that $z \geq 2n + 1$. Then there must be at least $2n+1$ occurrences of t, t^{-1} in w to reach index z, which contradicts the premise that w is of length $2n$. Therefore, $z \leq 2n$. □

A similar argument can be made considering the value of j_l.

Claim 2.4.11. $j_l = n$.

Proof. w Suppose to the contrary that $j_l < n$ or $j_l > n$.

- If $j_l < n$, then in w there must be at least n occurrences of t, t^{-1} to reach index $-n$, one occurrence of a to turn off the bulb at this index, and then at least $2n$ more occurrences of t, t^{-1} to reach index $z \geq 0$. This gives at least $2n+1$ generators in w, which contradicts the premise that w is of length $\leq 2n$. Therefore, $j_l \nless n$.
- If $j_l > n$, then in w there must be at least $n+1$ occurrences of t, t^{-1} to reach index $-j_l < -n$, one occurrence of a to turn on the bulb at this index, and then at least $n+1$ more occurrences of t, t^{-1} to reach index $z \geq 0$. This gives at least $2n+3$ generators in w, again contradicting the premise about the length of w. Therefore, $j_l \ngtr n$.

Therefore, by contradiction, $j_l = n$. We will use both of these claims to prove our third claim below. □

Claim 2.4.12. $i_k + |z - i_k| \leq 2n$.

Proof. We must consider if $i_k > n$ or $i_k \leq n$.

- If $i_k > n$, then in w there must be at least i_k occurrences of t, t^{-1} to reach index i_k so as to illuminate the bulb at that index. Furthermore, there must be least $|z - i_k|$ occurrences of t, t^{-1} to reach index z from i_k. Therefore, $i_k + |z - i_k| \leq \mathcal{O}_t(w)$. Note that as $w = x_\alpha \cdots x_1$, where $\alpha \leq 2n$, by Lemma 2.4.7, $\mathcal{O}t(w) + \mathcal{O}_a(w) \leq 2n$, giving that $\mathcal{O}_t(w) \leq 2n$, and thus, we have $i_k + |z - i_k| \leq 2n$ as desired.
- If $i_k \leq n$, then since $wd_n \notin H_n$ and $j_l = n$, it must be the case that $z > n$, giving $z > i_k$. Therefore $i_k + |z - i_k| = i_k + z - i_k = z$, and since $z \leq 2n$ from our previous claim, we have $i_k + |z - i_k| \leq 2n$ as desired.

□

Thus, we have $i_k + |z - i_k| \leq 2n$, which we will use to prove the desired result. Recall that we wish to show:

$$k + l + 2j_l + i_k + |z - i_k| \leq 6n + 1.$$

As $j_l = n$, it suffices to show

$$k + l + i_k + |z - i_k| \leq 4n + 1.$$

Suppose to the contrary that $k+l+i_k+|z-i_k| \geq 4n+2$. Then since $i_k+|z-i_k| \leq 2n$, we have $k+l \geq 2n+2$. In other words, $k+l = 2n+1+p$ for some $p \geq 1$. As there are $2n+1$ bulbs illuminated in d_n, this gives at least p additional bulbs being illuminated by w, and therefore $\mathcal{O}_a(w) \geq p$. This gives the following:

$$\begin{aligned} 2n \geq \mathcal{O}_t(w) + \mathcal{O}_a(w) \quad &\implies \quad 2n - \mathcal{O}_t(w) \geq \mathcal{O}_a(w) \\ &\implies \quad 2n - \mathcal{O}_t(w) \geq p \\ &\implies \quad 2n - p \geq \mathcal{O}_t(w) \end{aligned}$$

We have shown that $i_k + |z - i_k| \leq \mathcal{O}_t(w)$, and therefore, $i_k + |z - i_k| \leq 2n - p$. Furthermore, since $k + l = 2n + 1 + p$, we have

$$(k + l) + (i_k + |z - i_k|) \leq (2n + 1 + p) + 2n - p \leq 4n + 1$$

This contradicts our supposition above that $k + l + i_k + |z - i_k| \geq 4n + 2$. Therefore, such a supposition must be false and by contradiction our claim is proven. □

As such, we may not only say that The Lamplighter Group contains dead-end elements of at least n, but that these elements have depth $2n + 1$.

2.5 Another Perspective through Matrix Groups

In this section, we define yet another group that models the dynamics of the Lamplighter Group. We define its underlying set in the following definition.

Definition 2.5.1. ${L_2}^M = \left\{ \left(\begin{smallmatrix} t^k & P \\ 0 & 1 \end{smallmatrix}\right) | k \in \mathbb{Z},\ P \text{ a polynomial of } t, t^{-1} \right\}$

For operation amongst elements of this set, we employ standard matrix multiplication.

Having defined ${L_2}^M$, we next show that it is in fact a group.

Claim 2.5.2. ${L_2}^M$ is a group.

Proof. Of ${L_2}^M$, we must show closure, associativity, the existence of an identity element, and the existence of inverses.

To show ${L_2}^M$ is closed, let $l_1 = \left(\begin{smallmatrix} t^{k_1} & P_1 \\ 0 & 1 \end{smallmatrix}\right)$, $l_2 = \left(\begin{smallmatrix} t^{k_2} & P_2 \\ 0 & 1 \end{smallmatrix}\right) \in {L_2}^M$. We wish to show $l_1 l_2 \in {L_2}^M$. By the definition of matrix multiplication, we know

$$l_1 l_2 = \begin{pmatrix} t^{k_1} t^{k_2} & t^{k_1} P_2 + P_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{k_1+k_2} & t^{k_1} P_2 + P_1 \\ 0 & 1 \end{pmatrix}.$$

As $k_1 + k_2 \in \mathbb{Z}$ and $t^{k_1} P_2 + P_1$ is a polynomial of t, t^{-1}, we have that $l_1 l_2 \in {L_2}^M$.

To show associativity in ${L_2}^M$, let $l_1 = \left(\begin{smallmatrix} t^{k_1} & P_1 \\ 0 & 1 \end{smallmatrix}\right)$, $l_2 = \left(\begin{smallmatrix} t^{k_2} & P_2 \\ 0 & 1 \end{smallmatrix}\right)$, $l_3 = \left(\begin{smallmatrix} t^{k_3} & P_3 \\ 0 & 1 \end{smallmatrix}\right) \in {L_2}^M$. We wish to show $(l_1 l_2) l_3 = l_1 (l_2 l_3)$.

Note that

$$l_1 l_2 = \begin{pmatrix} t^{k_1+k_2} & t^{k_1} P_2 + P_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad l_2 l_3 = \begin{pmatrix} t^{k_2+k_3} & t^{k_2} P_3 + P_2 \\ 0 & 1 \end{pmatrix}.$$

Thus, we observe the following.

$$\begin{aligned}
(l_1 l_2) l_3 &= \begin{pmatrix} t^{k_1+k_2} & t^{k_1} P_2 + P_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{k_3} & P_3 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} t^{k_1+k_2+k_3} & P_1 + t^{k_1} P_2 + t^{k_1+k_2} P_3 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} t^{k_1} & P_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{k_2+k_3} & t^{k_2} P_3 + P_2 \\ 0 & 1 \end{pmatrix} \\
&= l_1 (l_2 l_3)
\end{aligned}$$

giving associativity as desired.

To show the existence of identity elements in ${L_2}^M$, note that $t^0 = 1$ and that the polynomial of t, t^{-1} in which all coefficients are 0 is 0, and thus, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in {L_2}^M$.

We know that such an element acts as an identity for any 2×2 matrix as in ${L_2}^M$, and thus that, for $\left(\begin{smallmatrix} t^k & P \\ 0 & 1 \end{smallmatrix}\right) \in {L_2}^M$,

$$\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}.$$

Lastly, to show the existence of inverses in ${L_2}^M$, let $l = \left(\begin{smallmatrix} t^k & P \\ 0 & 1 \end{smallmatrix}\right)$, $l' = \left(\begin{smallmatrix} t^{-k} & -t^{-k}P \\ 0 & 1 \end{smallmatrix}\right) \in {L_2}^M$. We wish to show $ll' = l'l = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

$$ll' = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-k} & -t^{-k}P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^k t^{-k} & -t^k t^{-k}P + P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$$

$$l'l = \begin{pmatrix} t^{-k} & -t^{-k}P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{-k}t^k & t^{-k}P - t^{-k}P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$$

Thus, we have shown that ${L_2}^M$ is a group. □

Recall ${L_2}' = \{\big((x_i), k\big) \mid x \in \oplus_{i=-\infty}^{\infty}(\mathbb{Z}_2)_i, k \in \mathbb{Z}\}$.

Theorem 2.5.3. [6, Lemma 0.2] ${L_2}' \cong {L_2}^M$.

Proof. Let ϕ be a function from ${L_2}'$ to ${L_2}^M$ such that, for $\big((x_i), k\big) \in {L_2}^M$,

$$\phi\Big(\big((x_i), k\big)\Big) = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$$

where $P = \sum_{i \in \mathbb{Z}} x_i \cdot t^i$. We claim that ϕ is an isomorphism.

We must show that ϕ is a homomorphism, surjective, and injective.

To show that ϕ is a homomorphism, let $l_1 = \big((x_i), k\big)$, $l_2 = \big((y_i), j\big) \in {L_2}'$. We wish to show $\phi(l_1 l_2) = \phi(l_1)\phi(l_2)$.

We note that $l_1 l_2 = \big((z_i), k + j\big)$ where $\forall i,\ z_i = x_i + y_{i-k}$. With this in mind, we first compute $\phi(l_1 l_2)$.

$$\phi(l_1 l_2) = \phi\Big(\big((z_i), k + j\big)\Big) = \begin{pmatrix} t^{k+j} & R \\ 0 & 1 \end{pmatrix} \text{ where } R = \sum_{i \in \mathbb{Z}} (x_i + y_{i-k}) \cdot t^i$$

We continue by computing $\phi(l_1)\phi(l_2)$.

$$\phi(l_1) = \phi\Big(\big((x_i), k\big)\Big) = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix} \text{ where } P = \sum_{i \in \mathbb{Z}} x_i \cdot t^i$$

$$\phi(l_2) = \phi\Big(\big((y_i), j\big)\Big) = \begin{pmatrix} t^j & Q \\ 0 & 1 \end{pmatrix} \text{ where } Q = \sum_{i \in \mathbb{Z}} y_i \cdot t^i$$

$$\phi(l_1)\phi(l_2) = \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^j & Q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{k+j} & P + t^k Q \\ 0 & 1 \end{pmatrix}$$

Thus, to show $\phi(l_1 l_2) = \phi(l_1)\phi(l_2)$, we must show that $R = P + t^k Q$. We do so below.

$$\begin{aligned}
P + t^k Q &= \sum_{i\in\mathbb{Z}} x_i \cdot t^i + t^k \cdot \sum_{h\in\mathbb{Z}} y_h \cdot t^h \\
&= \sum_{i\in\mathbb{Z}} x_i \cdot t^i + \sum_{h\in\mathbb{Z}} y_h \cdot t^{h+k} \\
&\quad \text{Let } h = i - k \\
&= \sum_{i\in\mathbb{Z}} x_i \cdot t^i + \sum_{i-k\in\mathbb{Z}} y_{i-k} \cdot t^i \\
&= \sum_{i\in\mathbb{Z}} x_i \cdot t^i + \sum_{i\in\mathbb{Z}} y_{i-k} \cdot t^i \\
&= \sum_{i\in\mathbb{Z}} (x_i + y_{i-k}) \cdot t^i \\
&= R
\end{aligned}$$

As $P + t^k Q = R$, we have shown that $\phi(l_1 l_2) = \phi(l_1)\phi(l_2)$ and therefore that ϕ is a homomorphism.

To show that ϕ is surjective, let $m = \left(\begin{smallmatrix} t^k & P \\ 0 & 1 \end{smallmatrix}\right) \in {L_2}^M$. We wish to show that $\exists l \in {L_2}'$ such that $\phi(l) = m$.

As P is a polynomial of t, t^{-1}, we let S be the set of integers i such that the coefficient of t^i in P is non-zero, giving $P = \sum_{s\in S} t^s$. We let l as follows

$$l = \big((x_i), k\big) \text{ where } x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{else} \end{cases}$$

Such an l is certainly in our domain ${L_2}'$, and furthermore

$$\phi(l) = \begin{pmatrix} t^k & R \\ 0 & 1 \end{pmatrix} \text{ where } R = \sum_{i\in\mathbb{Z}} x_i \cdot t^i$$

But since $\forall i,\ x_i \neq 0$ when $i \in S$, and therefore, $R = \sum_{s\in S} \cdot t^s$, giving $P = R$ and therefore $\phi(l) = m$ as desired.

To show that ϕ is injective, note that as ϕ is a homomorphism, we may show that ϕ cannot map a non-identity element in its domain to an identity element in its range.

Let $l = \big((x_i), k\big) \in L_2$ such that $\phi(l) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the identity in ${L_2}^M$. We wish to show that $l = \big((0_i), 0\big)$, the identity in ${L_2}'$.

We know that

$$\phi(l) = \phi\Big(\big((x_i), k\big)\Big) = \left(\begin{array}{cc} t^k & P \\ 0 & 1 \end{array} \right) \text{ where } P \text{ is a polynomial of } t, t^{-1}.$$

Furthermore as $\phi(l) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} t^k & P \\ 0 & 1 \end{smallmatrix}\right)$ we have that $t^k = 1$, which gives $k = 0$, and $P = 0$, which gives that $\forall i,\ x_i = 0$, or in other words, that $(x_i) = (0_i)$. Thus, we have $l = \big((x_i), k\big) = \big((0_i), 0\big)$ as desired.

We have shown that ϕ is an isomorphism and therefore that ${L_2}^M \cong {L_2}'$. $\square$

As ${L_2}^M \cong {L_2}' \cong L_2$, all of these groups successfully model the dynamics of the Lamplighter Group.

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