

3-2011

Strict Topologies On Spaces of Vector-valued Functions

Terje Høim

David A. Robbins

Trinity College, david.robbins@trincoll.edu

Follow this and additional works at: <http://digitalrepository.trincoll.edu/facpub>

 Part of the [Mathematics Commons](#)

Strict topologies on spaces of vector-valued functions

TERJE HÕIM AND D. A. ROBBINS

ABSTRACT. Let X be a completely regular Hausdorff space, and $\{E_x : x \in X\}$ a collection of non-trivial locally convex topological vector spaces indexed by X . Let $\mathcal{E} = \dot{\bigcup}_{x \in X} E_x$ be their disjoint union. We investigate a species of strict topology on a vector space \mathcal{F} of choice functions $\sigma : X \rightarrow \mathcal{E}$ ($\sigma(x) \in E_x$), and obtain Stone–Weierstrass and spectral synthesis analogues. We also obtain completeness results in some special cases.

1. Introduction

Let X be a completely regular Hausdorff space, and let E be a topological vector space. The space $C(X, E)$ of continuous functions from X to E can be topologized in a variety of ways, see e.g. the paper [10] by L.A. Khan. Our goal in this paper is to extend, as far as we can, results of Khan and others to the situation where the continuous functions on X take their values in spaces which may vary with $x \in X$. We will use methods from the theory of bundles of topological vector spaces.

By X we will always mean a Hausdorff topological space; unless otherwise specified, X will also be completely regular. We will say that a complex-valued function a on X vanishes at infinity if for each $\varepsilon > 0$ there exists a compact $K \subset X$ such that $|a(x)| < \varepsilon$ whenever $x \notin K$. (Clearly, if such a function a is continuous (or if $x \mapsto |a(x)|$ is upper semicontinuous), it is bounded.) If g is any function defined on X , and if $C \subset X$, we denote by g_C the restriction of g to C . If G is a collection of functions on X , set $G_C = \{g_C : g \in G\}$. If Y is a topological space, then \overline{W} will denote the closure of $W \subset Y$.

Received November 14, 2010.

2010 *Mathematics Subject Classification.* 46A05, 46H25.

Key words and phrases. Strict topology, bundles of topological vector spaces.

We consider the following situation, which parallels that discussed in [9], and which we repeat here for convenience: let $\{E_x : x \in X\}$ be a collection of non-trivial Hausdorff topological vector spaces (over the same ground field, \mathbb{R} or \mathbb{C}), indexed by the completely regular space X . Let

$$\mathcal{E} = \dot{\bigcup}\{E_x : x \in X\}$$

be the disjoint union of the E_x , and let $\pi : \mathcal{E} \rightarrow X$ be the natural projection. Assume further that on \mathcal{E} we have a set \mathfrak{W} of functions w such that for each $x \in X$, $\{w^x : w \in \mathfrak{W}\}$ is a family of seminorms which generates the topology on E_x (where w^x is the restriction of w to E_x). Finally, let \mathcal{F} be a vector space of choice functions $\sigma : X \rightarrow \mathcal{E}$ (i.e. $\sigma(x) \in E_x$ for all x) such that the following conditions hold:

C1) for each $x \in X$, $\phi_x(\mathcal{F}) = \{\sigma(x) : \sigma \in \mathcal{F}\} = E_x$ (in this case, \mathcal{F} is said to be full; ϕ_x is the evaluation map at x);

C2) \mathcal{F} is a $C_b(X)$ -module;

C3) for each $\sigma \in \mathcal{F}$ and $w \in \mathfrak{W}$, the numerical function $x \mapsto w^x(\sigma(x))$ is upper semicontinuous and bounded on X ; and

C4) \mathcal{F} is closed in the seminorm topology determined by \mathfrak{W} , as described below.

For each $w \in \mathfrak{W}$, we define a seminorm \hat{w} on \mathcal{F} by

$$\hat{w}(\sigma) = \sup_{x \in X} \{w^x(\sigma(x))\}.$$

Since the map $x \mapsto w^x(\sigma(x))$ is upper semicontinuous and bounded, the \hat{w} ($w \in \mathfrak{W}$) determine a locally convex topology on \mathcal{F} , where the sets

$$N(\sigma, \hat{w}, \varepsilon) = \{\tau \in \mathcal{F} : \hat{w}(\sigma - \tau) < \varepsilon\}$$

form a subbasis for this topology on \mathcal{F} , as $w \in \mathfrak{W}$ and $\varepsilon > 0$ vary. As in [9] it can be easily checked that the multiplication $(a, \sigma) \mapsto a\sigma$ is jointly continuous from $C_b(X) \times \mathcal{F}$ to \mathcal{F} (where $C_b(X)$ is given its sup-norm topology), so that \mathcal{F} is a topological $C_b(X)$ -module.

The topology on \mathcal{F} is thus that of uniform convergence with respect to each of the seminorms \hat{w} ($w \in \mathfrak{W}$); given \mathcal{F} and \mathfrak{W} , we denote the space by (\mathcal{F}, u) .

We note that these functions w ($w \in \mathfrak{W}$) also determine a topology on the fibered space \mathcal{E} . Here, a neighborhood around $z \in E_x \subset \mathcal{E}$ is given by tubes of the form

$$T(U, \sigma, w, \varepsilon) = \{z' \in \mathcal{E} : w^{\pi(z')}(\sigma(\pi(z'))) - z' < \varepsilon \text{ and } \pi(z') \in U\},$$

where U is an open neighborhood of x , $w \in \mathfrak{W}$, $\varepsilon > 0$, and $\sigma \in \mathcal{F}$ is any element such that $\sigma(x) = z$.

By [4, Proposition 5.8], (\mathcal{F}, u) is then topologically and algebraically isomorphic to a subspace of the section space $\Gamma(\pi)$ of sections of a bundle

$\pi : \mathcal{E} \rightarrow X$ of topological vector spaces. (A section is a continuous choice function.) Moreover, for each compact $K \subset X$, $\mathcal{F}_K = \{\sigma_K : \sigma \in \mathcal{F}\}$, is by [4, Theorem 5.9] algebraically and topologically isomorphic to the space $\Gamma(\pi_K)$ of sections of the restriction bundle $\pi_K : \mathcal{E}_K \rightarrow K$. We refer the reader to [4, Chapter 5] for details of the construction of such bundles. Intuitively, if $\sigma \in \mathcal{F}$, we can think of $\sigma(x)$ as moving continuously through the spaces E_x as x moves continuously through X . (See also the introduction to [8] and various of its references, especially [6].) If $\pi : \mathcal{E} \rightarrow X$ is the bundle determined by \mathcal{F} and \mathfrak{W} , we define $\Gamma_b(\pi)$, the space of bounded sections of the bundle, by

$$\Gamma_b(\pi) = \{\sigma \in \Gamma(\pi) : \widehat{w}(\sigma) < \infty \text{ for all } w \in \mathfrak{W}\}.$$

In accord with the above notation, we denote $\Gamma_b(\pi)$ under the bundle topology determined by \mathcal{F} and \mathfrak{W} as $(\Gamma_b(\pi), u)$. Thus, a net $(\sigma_\lambda) \subset \Gamma_b(\pi)$ converges to $\sigma \in \Gamma_b(\pi)$ in the u -topology if and only if for each $\varepsilon > 0$ and $w \in \mathfrak{W}$ we have $\widehat{w}(\sigma_\lambda - \sigma) < \varepsilon$ (i.e. $\sigma_\lambda \in N(\sigma, \widehat{w}, \varepsilon)$) if λ is sufficiently large. Note that, as usual, we may think of \mathfrak{W} as being a cone: if $\alpha > 0$ and $w, w' \in \mathfrak{W}$, then $\alpha w + w' \in \mathfrak{W}$, where $(\alpha w + w')^x(z) = \alpha w^x(z) + w'^x(z)$, for $z \in E_x \subset \mathcal{E}$.

Now, let \mathcal{S}_0 be the set of non-negative upper semicontinuous functions on X which vanish at infinity. Certainly \mathcal{S}_0 is closed under addition and multiplication. Let

$$\mathfrak{V} = \mathcal{S}_0 \cdot \mathfrak{W} = \{aw : a \in \mathcal{S}_0; w \in \mathfrak{W}\},$$

where aw is defined by $(aw)^x = a(x)w^x$. Define

$$\widehat{aw}(\sigma) = \sup_{x \in X} \{a(x)w^x(\sigma(x))\} \leq \|a\| \widehat{w}(\sigma).$$

It is then easily checked that seminorms of the form $\widehat{aw} \in \mathfrak{V}$ generate a locally convex linear topology on $\Gamma_b(\pi)$ (and hence on \mathcal{F}), and that the numerical function $x \mapsto v^x(\sigma(x)) = a(x)w^x(\sigma(x))$ is upper semicontinuous and vanishes at infinity on X for each $\sigma \in \Gamma_b(\pi)$. We call this locally convex topology on $\Gamma_b(\pi)$ the strict topology, and denote the space by $(\Gamma_b(\pi), \beta)$. We may consider \mathfrak{V} also to be a cone.

Continuing, let \mathfrak{K} be a compact cover of X which is closed under finite unions. Given \mathcal{F} and \mathfrak{W} , we note that χ_K , the characteristic function of $K \in \mathfrak{K}$, is in \mathcal{S}_0 . We can then define the topology of compact convergence in $\Gamma_b(\pi)$ with respect to \mathfrak{K} via the seminorms

$$(\widehat{\chi_K w})(\sigma) = \widehat{w}_K(\sigma) = \sup_{x \in K} w^x(\sigma(x)) = \sup_{x \in K} \{\chi_K(x)w^x(\sigma(x))\},$$

where $K \in \mathfrak{K}$ and $w \in \mathfrak{W}$. Again, these seminorms describe a locally convex linear topology on $\Gamma_b(\pi)$.

Two particular instances of \mathfrak{K} are worth noting: if \mathfrak{K} is the collection of all compact subsets of X , we denote the resulting topology on $\Gamma_b(\pi)$ by κ , which is the topology of compact convergence in $\Gamma_b(\pi)$ with respect to \mathfrak{W} .

If \mathfrak{K} is the collection of all finite subsets of X , the resulting topology is that of pointwise convergence, and denoted by p .

It is evident that for a given \mathcal{F} and \mathfrak{W} , u -convergence in $\Gamma_b(\pi)$ implies β -convergence, which in turn implies κ -convergence, which finally implies p -convergence.

It is time for some examples, which are analogous to those in [9].

1) If E is a locally convex topological vector space, whose topology is generated by seminorms $\psi \in \Psi$, then $\mathcal{F} = C_b(X, E)$ is the space of bounded continuous functions $\sigma : X \rightarrow E$. It is a subspace of $C(X, E)$, the space of all such continuous functions, which we can identify with the space of sections of the trivial bundle $\pi : \mathcal{E} \rightarrow X$, where $E_x = E$ for all $x \in X$ and the topology on \mathcal{E} is homeomorphic to $X \times E$ (if $z \in E_x \subset \mathcal{E}$, then $z \leftrightarrow (x, z) \in X \times E$). This is the situation studied in [10].

2) For each $x \in X = \mathbb{R}$, let $E_x = \mathbb{C}$, and let $\mathcal{F} = c_0(X)$ be the closure in the sup norm of the set of functions with finite support. Here, \mathfrak{W} is a singleton ($\mathfrak{W} = \{|\cdot|\}$), and \mathcal{F} is u -closed, hence u -complete, in $\Gamma_b(\pi)$, but not β -closed; see [9] for details.

3) If X is completely regular, and if each E_x is a Banach space, and if $\pi : \mathcal{E} \rightarrow X$ is a bundle of Banach spaces in the sense of [4] and other references below, then $\mathfrak{W} = \{\|\cdot\|\}$, and \mathfrak{W} gives us the strict topology on $\Gamma_b(\pi)$, the space of bounded sections of the bundle.

4) If X is completely regular, let $\mathcal{F} = C_b(X)$. The family of seminorms is $\mathfrak{W} = \{|\cdot|\}$, each $E_x = \mathbb{C}$, and the strict topology β defined by $\mathfrak{W} = \mathcal{S}_0 \cdot \mathfrak{W}$ is among those which have played a part in studying $C_b(X)$ with the strict topology.

With these examples in mind, we can put the current situation in a larger context. Given \mathcal{F} and \mathfrak{W} which satisfy conditions C1) – C4), then as we have seen they determine a bundle $\pi : \mathcal{E} \rightarrow X$ of topological vector spaces, with a section space $\Gamma(\pi)$. The constant-fiber instance of this general situation is the space $C(X, E)$ of all continuous functions from X to the (constant) locally convex space E , in the following way. Suppose that the topology of E is determined by a family Ψ of seminorms, and that $C_b(X, E) = \{\sigma \in C(X, E) : \psi(\sigma(x)) < \infty \text{ for all } \psi \in \Psi\}$. Then the family of seminorms $\widehat{\Psi}$ on $\mathcal{E} = \bigcup_{x \in X} E_x$, where for $\psi \in \Psi$ we define $\widehat{\psi} \in \widehat{\Psi}$ by

$$\widehat{\psi}(\sigma) = \sup_{x \in X} \psi^x(\sigma(x)) = \sup_{x \in X} \psi(\sigma(x))$$

for $\psi \in \Psi$ and $\sigma \in C_b(X, E)$, together with $C_b(X, E)$, satisfy conditions C1) – C4). As in [10], we can then let $\mathcal{S} \subset \mathcal{B}(X)$ (where $\mathcal{B}(X)$ is the space of bounded real functions on X), be a cone of non-negative functions on X , and consider the resulting topologies on $C(X, E)$ or its subspaces generated by $\mathcal{S} \cdot \Psi$. So, for example, if \mathcal{S} is the space of non-negative upper semicontinuous

functions which vanish at infinity (i.e. $\mathcal{S} = \mathcal{S}_0$), we get the strict topology on $\Gamma_b(\pi)$, as defined here. Or, if \mathcal{S} is the space of functions with compact support, we get the κ -topology. Along these lines, as in [9], we point out that several definitions for the strict topology β on $C_b(X)$ have been investigated over the years: e.g. in [3] X is assumed to be locally compact and \mathcal{S} is taken to be $C_0(X)$, the space of continuous functions on X which vanish at infinity. In [2] X is completely regular and \mathcal{S} is the space of non-negative bounded functions which vanish at infinity (again, our \mathcal{S}_0). In [1] and [15] \mathcal{S} is the space of positive upper semicontinuous functions vanishing at infinity, and it is noted that for $C_b(X)$ this is equivalent to using the bounded positive functions which vanish at infinity. See also [10] in this regard. Because upper semicontinuity of the seminorms is crucial in the development of our results, we will use \mathcal{S}_0 in order to retain the upper semicontinuity of functions in \mathfrak{W} . It is the case, however, that in the end, as with $C(X, E)$, we could eliminate upper semicontinuity, as the following brief development shows.

Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and let $\pi : \mathcal{E} \rightarrow X$ be the bundle determined by \mathcal{F} and \mathfrak{W} . Let $\mathcal{S} \subset \mathcal{B}(X)$ be any cone of non-negative bounded functions on X , and suppose that for no $x \in X$ does $a(x) = 0$ for all $a \in \mathcal{S}$. Then $\mathcal{S} \cdot \mathfrak{W}$ determines a topology on $\Gamma_b(\pi)$ in the evident fashion. (That the functions in \mathcal{S} do not all vanish at any $x \in X$ allows us to maintain the condition that topology on each E_x is determined by $(\mathcal{S} \cdot \mathfrak{W})^x = \{a(x)w^x : a \in \mathcal{S}; w \in \mathfrak{W}\}$.) Suppose that $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{B}(X)$ are two such cones. Evidently, the topology on $\Gamma_b(\pi)$ determined by $\mathcal{S}_1 \cdot \mathfrak{W}$ is weaker than the topology on $\Gamma_b(\pi)$ determined by $\mathcal{S}_2 \cdot \mathfrak{W}$; i.e. convergence of a net $(\sigma_\lambda) \subset \Gamma_b(\pi)$ with respect to $\mathcal{S}_2 \cdot \mathfrak{W}$ implies convergence of the net with respect to $\mathcal{S}_1 \cdot \mathfrak{W}$.

The following Lemma is based on [10, Lemma 3.2].

Lemma 1. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{B}(X)$ be sets of non-negative functions. Suppose that for each $a_1 \in \mathcal{S}_1$ there exist $\delta > 0$ and $a_2 \in \mathcal{S}_2$ such that $a_1(x) < \delta a_2(x)$ for all $x \in X$. Then the $\mathcal{S}_1 \cdot \mathfrak{W}$ topology on $\Gamma_b(\pi)$ is weaker than the $\mathcal{S}_2 \cdot \mathfrak{W}$ topology on $\Gamma_b(\pi)$.*

Proof. A typical subbasic $\mathcal{S}_1 \cdot \mathfrak{W}$ -neighborhood of 0 in $\Gamma_b(\pi)$ is of the form $N(0, \widehat{a_1 w}, \varepsilon)$, where $a_1 w \in \mathcal{S}_1 \cdot \mathfrak{W}$ and $\varepsilon > 0$. Choose $\delta > 0$ and $a_2 \in \mathcal{S}_2$ such that $a_1(x) < \delta a_2(x)$ for all $x \in X$, and consider the $\mathcal{S}_2 \cdot \mathfrak{W}$ -neighborhood of 0 given by $N(0, \widehat{a_2 w}, \varepsilon/\delta)$. If $\sigma \in N(0, \widehat{a_2 w}, \varepsilon/\delta)$, then for all $x \in X$ we have

$$a_1(x)w^x(\sigma(x)) \leq \delta a_2(x)w^x(\sigma(x)) < \varepsilon,$$

so that $\widehat{a_1 w}(\sigma) < \varepsilon$, and thus $\sigma \in N(0, \widehat{a_1 w}, \varepsilon)$. As a consequence, any $\mathcal{S}_1 \cdot \mathfrak{W}$ -neighborhood of 0 contains an $\mathcal{S}_2 \cdot \mathfrak{W}$ -neighborhood of 0, so that the $\mathcal{S}_1 \cdot \mathfrak{W}$ topology is weaker than the $\mathcal{S}_2 \cdot \mathfrak{W}$ topology. \square

Lemma 2. *Let $a_1 \in \mathcal{B}_0(X)$, the space of bounded non-negative functions on X which vanish at infinity. Then there exists an upper semicontinuous*

function a_2 on X , vanishing at infinity, and $\delta > 0$, such that $a_1(x) < \delta a_2(x)$ for all $x \in X$.

Proof. See the proof of [10, Theorem 3.7]. \square

Corollary 3. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4). Let \mathcal{S}_0 be the cone of non-negative upper semicontinuous functions on X which vanish at infinity. Then the $\mathcal{S}_0 \cdot \mathfrak{W}$ - and $\mathcal{B}_0(X) \cdot \mathfrak{W}$ -topologies on $\Gamma_b(\pi)$ are equivalent.*

Proof. Apply the remark preceding Lemma 1, and then Lemmas 1 and 2. \square

Corollary 4. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and suppose that X is compact. Then the u -, β -, and κ -topologies coincide on $\Gamma(\pi) = \Gamma_b(\pi)$.*

Proof. We have already noted that $\kappa \subset \beta \subset u$. It suffices to show that $u \subset \kappa$; but for this, note that $\chi_X = 1$. I.e. $\{\chi_X\} \cdot \mathfrak{W} = \mathfrak{W} \subset \mathcal{S} \cdot \mathfrak{W}$, where \mathcal{S} is the set of non-negative upper semicontinuous functions with compact support. \square

2. Stone–Weierstrass and spectral synthesis results

We first prove some general results which have analogies in the theory of bundles of Banach spaces.

Lemma 5. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and suppose that $K \subset X$ is compact. Suppose that $M \subset \mathcal{F}$ is a $C_b(X)$ -submodule of \mathcal{F} such that $M_x = \phi_x(M) = \{\sigma(x) : \sigma \in M\}$ is dense in E_x for each $x \in K$. Then $M_K = \{\sigma_K : \sigma \in M\}$ is dense in $\mathcal{F}_K = \{\tau_K : \tau \in \mathcal{F}\}$ in the β_K -topology determined by $\mathfrak{V}_K = \{x \mapsto v^x(\cdot) : v \in \mathfrak{V}, x \in K\}$.*

Proof. This is a variant of [4, Theorem 4.2]. Suppose that $v \in \mathfrak{V}$ and $\varepsilon > 0$ are given. Let $\sigma \in \mathcal{F}$. We will construct $\tau \in M$ such that $v^y(\tau(y) - \sigma(y)) < \varepsilon$ for all $y \in K$.

Let $x \in K$. Since M_x is dense in E_x , there exists $z \in M_x$ such that $v^x(z - \sigma(x)) < \varepsilon$. Consequently, there exists $\tau_x \in M$ such that $\tau_x(x) = z$. For each $x \in X$, choose a neighborhood U_x of x such that if $y \in U_x$, then $v^y(\tau_x(y) - \sigma(y)) < \varepsilon$. Choose a finite subcover $U_{x_j} = U_j$ ($j = 1, \dots, k$) from among the U_x . From [2, Lemma 2.1] we can choose continuous functions $a_j : X \rightarrow [0, 1]$ such that 1) each a_j is supported on U_j ; 2) $\sum_{j=1}^k a_j(x) = 1$ for $x \in K$; and 3) $\sum_{j=1}^k a_j(x) \leq 1$ for all $x \in X$. Set $\tau = \sum_{j=1}^k a_j \tau_{x_j} \in M$. Then it is easily checked that $v^y(\tau(y) - \sigma(y)) < \varepsilon$ for all $y \in K$, i.e. $\widehat{v}(\tau - \sigma) < \varepsilon$. \square

Proposition 6. *Suppose that \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above. Suppose also that $M \subset \mathcal{F}$ is a $C_b(X)$ -submodule of \mathcal{F} such that $M_x =$*

$\phi_x(M) = \{\sigma(x) : \sigma \in M\}$ is dense in E_x for each $x \in X$. Then M is β -dense in \mathcal{F} .

Proof. Let $v \in \mathfrak{V}$ and $\varepsilon > 0$ be given, and let $\sigma \in \mathcal{F}$. Then there exists compact $K \subset X$ such that $v^y(\sigma(y)) < \varepsilon$ whenever $y \notin K$. By the preceding Lemma, there exists $\tau \in M$ such that $v^y(\tau(y) - \sigma(y)) < \varepsilon$ for all $y \in K$. By the upper semicontinuity, we can choose a neighborhood U of K such that in fact $v^y(\tau(y) - \sigma(y)) < \varepsilon$ for all $y \in U$. Since X is completely regular, we can also choose a continuous $a : X \rightarrow [0, 1]$ such that $a(K) = 1$ and $a(X \setminus U) = 0$. Then $a\tau \in M$, and it can be checked that $v^y((a\tau)(y) - \sigma(y)) < \varepsilon$ for all $y \in X$; i.e. that $\widehat{v}(a\tau - \sigma) < \varepsilon$. \square

Note that this Stone–Weierstrass result is an extension of those to be found in e.g. [16, Theorem 3] and [10].

Corollary 7 (See [16], Section 3). *Suppose that \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above, with $E_x = E$ for some fixed topological vector space E . Let $\mathcal{F}' \subset \mathcal{F}$ be a $C_b(X)$ -submodule of \mathcal{F} such that for each $x \in X$ and each closed subspace $T \subset E$ of codimension 1, there exists $\sigma \in \mathcal{F}'$ such that $\sigma(x) \notin T$. Then \mathcal{F}' is β -dense in \mathcal{F} .*

Proof. We need only show that $\mathcal{F}'_x = \phi_x(\mathcal{F}')$ is dense in E for each $x \in X$. If not, then for some $x \in X$, $\overline{\mathcal{F}'_x} \subset E$ is a proper closed subspace, and so for a fixed $z \in E \setminus \overline{\mathcal{F}'_x}$, by the Hahn–Banach theorem there is a continuous functional $f \in E^*$ (the continuous dual of E) such that $\overline{\mathcal{F}'_x} \subset \ker f$ and $f(z) = 1$. But $\ker f$ is a closed subspace of codimension 1, and so there exists $\sigma \in \mathcal{F}'$ such that $z = \sigma(x) \notin \ker f$, and hence $\sigma(x) \notin \mathcal{F}'_x$, a contradiction. \square

We can also prove a spectral synthesis-type result extending that in [13], and analogous to that in [9, Proposition 10].

Suppose that \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above. It is easy to see that the evaluation map $\phi_x : \mathcal{F} \rightarrow E_x$, $\sigma \mapsto \sigma(x)$, is continuous. Hence, if $f_x \in E_x^*$, the composition $f_x \circ \phi_x$ is also continuous. Note that for $\sigma \in \mathcal{F}$ and $a \in C_b(X)$, we have $(f_x \circ \phi_x)(a\sigma) = a(x)f_x(\sigma(x))$. From this, it follows that $\ker(f_x \circ \phi_x)$ is a closed submodule of \mathcal{F} of codimension 1; i.e. a maximal closed submodule. It turns out that all closed submodules of \mathcal{F} of codimension 1 arise in this fashion.

Lemma 8. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above, and suppose that $M \subset \mathcal{F}$ is a β -closed proper submodule. Then there exists $x \in X$ such that $\overline{\phi_x(M)} = \overline{M_x}$ is a closed proper subspace of E_x .*

Proof. Clearly, M_x is a subspace of E_x , and hence so is $\overline{M_x}$. Suppose, if possible, that for all $x \in X$ we have $\overline{M_x} = E_x$. But now M satisfies the conditions of Proposition 6 above, so that M is dense in \mathcal{F} ; since M is closed, this forces $M = \mathcal{F}$, a contradiction. \square

Corollary 9. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above, and suppose that $0 \neq g \in \mathcal{F}^*$, with $M = \ker g$ a submodule of \mathcal{F} . Then there exists $x \in X$ such that $\overline{M_x}$ is a proper subspace of E_x .*

Proposition 10. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above, and suppose that $g \in \mathcal{F}^*$, and that $M = \ker g$ is a submodule of \mathcal{F} . Then there exist unique $x \in X$ and $f_x \in E_x^*$ such that $g = f_x \circ \phi_x$.*

Proof. Since M is a closed proper submodule of \mathcal{F} , there exists $x \in X$ such that $\overline{M_x}$ is a closed proper subspace of E_x . Then $\frac{E_x}{\overline{M_x}} \neq 0$. The evaluation $\phi_x : \mathcal{F} \rightarrow E_x$ maps M into M_x , so there is a unique linear map $\xi : \frac{\mathcal{F}}{M} \rightarrow \frac{E_x}{\overline{M_x}}$ which makes this diagram commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi_x} & E_x \\ \delta \downarrow & & \downarrow \delta_x \\ \frac{\mathcal{F}}{M} & \xrightarrow{\xi} & \frac{E_x}{\overline{M_x}} \end{array}$$

(where δ and δ_x are the quotient maps). Since $\phi_x : \mathcal{F} \rightarrow E_x$ is surjective (recall that \mathcal{F} was chosen to be full), so is ξ . But $\frac{\mathcal{F}}{M}$ is one-dimensional, and as a consequence so also is $\frac{E_x}{\overline{M_x}}$. In particular, then, $\overline{M_x}$ is a closed subspace of codimension 1, and so it is the kernel of some non-zero functional $f_x \in E_x^*$, i.e. $\overline{M_x} = \ker f_x$. Clearly, $f_x \circ \phi_x$ is non-trivial. Moreover, if $\sigma \in M$, then $\sigma(x) \in M_x \subset \overline{M_x}$, and so $(f_x \circ \phi_x)(\sigma) = 0$; therefore $M = \ker g \subset \ker (f_x \circ \phi_x)$. But since both M and $\ker (f_x \circ \phi_x)$ are closed and maximal, this forces $M = \ker (f_x \circ \phi_x)$. Thus, there exists $\alpha \neq 0$ such that $g = \alpha (f_x \circ \phi_x) = (\alpha f_x) \circ \phi_x$; since $\alpha f_x \in E_x^*$, we are done. \square

For a subspace $\mathcal{F}' \subset \mathcal{F}$ and $x \in X$, set $\mathcal{F}'^x = \{\sigma \in \mathcal{F} : \sigma(x) \in \overline{\mathcal{F}'_x}\}$. Then \mathcal{F}'^x is a β -closed submodule of \mathcal{F} . [For, if $\sigma \in \mathcal{F}$ and $a \in C_b(X)$, we have $(a\sigma)(x) = a(x)\sigma(x) \in \overline{\mathcal{F}'_x}$ whenever $\sigma(x) \in \overline{\mathcal{F}'_x}$; i.e. $a\sigma \in \mathcal{F}'^x$ whenever $\sigma \in \mathcal{F}'^x$. Moreover, fix $x \in X$. If τ is in the β -closure of \mathcal{F}'^x , then for each $v \in \mathfrak{V}$ and $\varepsilon > 0$, there exists $\sigma \in \mathcal{F}'^x$ such that $\widehat{v}(\tau - \sigma) < \varepsilon$; in particular $v^x(\tau(x) - \sigma(x)) < \varepsilon$. But $\sigma(x) \in \overline{\mathcal{F}'_x}$, and hence so does $\tau(x) \in \overline{\mathcal{F}'_x}$. Thus $\tau \in \mathcal{F}'^x$.]

We use this remark to prove the following.

Corollary 11. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4) above, and suppose that $M \subset \mathcal{F}$ is a β -closed submodule. Then $M = \bigcap_{x \in X} (\overline{M_x})^x$.*

Proof. Set $M' = \bigcap_{x \in X} (\overline{M_x})^x$. Since it is clear that $M \subset M'$, we claim that $M' \subset M$.

As noted immediately above, each $(\overline{M_x})^x$ is a closed submodule of \mathcal{F} . Let $\sigma \in M', v \in \mathfrak{V}$, and $\varepsilon > 0$. Choose compact $K \subset X$ such that $v^x(\sigma(x)) < \frac{\varepsilon}{2} = \varepsilon'$ if $x \notin K$. For each $x \in K$, since $\sigma(x) \in \overline{M_x}$, there exists $\tau_x \in M$ such that $v^y(\sigma(x) - \tau_x(x)) < \varepsilon'$. From the upper semicontinuity of the seminorms, there exists for each $x \in K$ an open neighborhood U_x of x such that if $y \in U_x$ then $v^y(\sigma(y) - \tau_x(y)) < \varepsilon'$. Choose a finite subcover $U_{x_j} = U_j$ of K . Again using [2, Lemma 1], we can choose continuous functions $a_j : X \rightarrow [0, 1]$ ($k = 1, \dots, n$) such that 1) $\sum_{j=1}^n a_j(y) = 1$ for $y \in K$; 2) a_j is supported on U_j for each $j = 1, \dots, n$; and 3) $\sum_{j=1}^n a_j(y) \leq 1$ for each $y \in X$.

Let $\tau = \sum_{j=1}^n a_j \tau_{x_j}$, and let $y \in X$. There are the following three possibilities.

1) If $y \in K$, then

$$\begin{aligned} v^y(\sigma(y) - \tau(y)) &= v^y \left(\sum_{j=1}^n a_j(y)(\sigma(y) - \tau_{x_j}(y)) \right) \\ &\leq \sum_{j=1}^n a_j(y) v^y(\sigma(y) - \tau_{x_j}(y)) \\ &< \varepsilon. \end{aligned}$$

2) If $y \in \bigcup_{j=1}^n U_j \setminus K$, then

$$\begin{aligned} v^y(\sigma(y) - \tau(y)) &= v^y \left(\sum_{j=1}^n a_j(y)(\sigma(y) - \tau_{x_j}(y)) + \left(1 - \sum_{j=1}^n a_j(y) \right) \sigma(y) \right) \\ &\leq v^y \left(\sum_{j=1}^n a_j(y)(\sigma(y) - \tau_{x_j}(y)) \right) \\ &\quad + \left(1 - \sum_{j=1}^n a_j(y) \right) v^y(\sigma(y)) \\ &< \varepsilon' + \varepsilon' \\ &< \varepsilon \end{aligned}$$

(because $y \in U_j$ for some j and $\sum_{j=1}^n a_j(y) \leq 1$, but $y \notin K$).

3) If $y \in X \setminus \bigcup_{j=1}^n U_j$, then

$$\begin{aligned} v^y(\sigma(y) - \tau(y)) &= v^y \left(\sigma(y) - \sum_{j=1}^n a_j(y) \tau_{x_j}(y) \right) \\ &= v^y(\sigma(y)) \\ &< \varepsilon' \\ &< \varepsilon \end{aligned}$$

(since $a(y) = 0$ for all $y \notin \bigcup_{j=1}^n U_j$).

Thus, for all $y \in X$, we have $v^y(\sigma(y) - \tau(y)) < \varepsilon$, and so $\tau \in N(\sigma, \hat{v}, \varepsilon)$. This forces $M' \subset M$. \square

Proposition 12. *Let \mathcal{F} and \mathfrak{M} satisfy conditions C1) – C4) above, and suppose that $M \subset \mathcal{F}$ is a β -closed submodule. Then M is the intersection of all β -closed maximal submodules which contain it.*

Proof. Our proof is obtained by translating the language of [9, Proposition 10] into our current situation.

Set $P = \{x \in X : \overline{M_x} \text{ is a proper subspace of } E_x\}$. Note in general that for $g \in \mathcal{F}^*$, with $g = f_x \circ \phi_x$ for $f_x \in E_x^*$ and $x \in P$, we have the following: $M \subset \ker g$, if and only if $M_x \subset \ker f_x$, if and only if $\overline{M_x} \subset \ker f_x$, if and only if $(\overline{M_x})^x \subset \ker(f_x \circ \phi_x)$. But when $x \in P$, $\overline{M_x} \subset E_x$ is a subspace, and so from the Hahn–Banach theorem we have $\overline{M_x} = \bigcap \{\ker f_x : f_x \in E_x^* \text{ and } \overline{M_x} \subset \ker f_x\}$. Thus, if $x \in P$, we have

$$\begin{aligned} (\overline{M_x})^x &= \bigcap \{\ker(f_x \circ \phi_x) : f_x \in E_x^* \text{ and } \overline{M_x} \subset \ker f_x\} \\ &= \bigcap \{\ker g : g = f_x \circ \phi_x \in \mathcal{F}^* \text{ and } (\overline{M_x})^x \subset \ker g\}. \end{aligned}$$

Finally,

$$\begin{aligned} M &= \bigcap_{x \in X} (\overline{M_x})^x = \bigcap_{x \in P} (\overline{M_x})^x \\ &= \bigcap_{x \in P} \{\ker g : g = f_x \circ \phi_x \in \mathcal{F}^* \text{ and } (\overline{M_x})^x \subset \ker g\} \\ &= \bigcap_{x \in X} \{\ker g : g = f_x \circ \phi_x \in \mathcal{F}^* \text{ and } M \subset \ker g\}. \end{aligned}$$

\square

This result is analogous to [16, Theorem 5].

If X is compact, the above result translates into the following well-known fact that $C(X)$ satisfies spectral synthesis.

Corollary 13. *Let X be compact. Then $C(X)$ satisfies spectral synthesis; i.e. each closed ideal in $C(X)$ is the intersection of the closed maximal ideals which contain it.*

Proof. A $C(X)$ -submodule of $C(X)$ is an ideal. □

The corollary above points out the necessity in Proposition 12 that M be a submodule: if, for example, $X = [0, 1]$, then for the Lebesgue measure $\mu \in C(X)^*$, $\ker \mu$ is obviously a closed maximal subspace, but is not contained in any closed maximal ideal (equivalently in this situation, closed maximal submodule).

(See also [13], where this spectral synthesis result is proved for section spaces $\Gamma(\pi)$, where $\pi : \mathcal{E} \rightarrow X$ is a Banach bundle over the compact base space X .)

3. Completeness and other results for strict topologies

We now consider various questions regarding strict topologies on spaces of vector-valued functions. Recall from the Introduction that if \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), then there is a bundle $\pi : \mathcal{E} \rightarrow X$, with fibers E_x , such that \mathcal{F} is a subspace of the space $\Gamma_b(\pi)$ of bounded sections, which in turn is a subspace of $\Gamma(\pi)$, the space of all sections of π .

Our first results involve completeness.

Proposition 14. *Let X be locally compact, and let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4). Suppose that for each $x \in X$, E_x is complete, and let $\Gamma_b(\pi)$ be the space of bounded sections of the bundle $\pi : \mathcal{E} \rightarrow X$ whose topology is determined by \mathcal{F} and \mathfrak{W} . Then $(\Gamma_b(\pi), \beta)$ is complete.*

Proof. Let (σ_λ) be a Cauchy net in $(\Gamma_b(\pi), \beta)$. Then given $\varepsilon > 0$ and $v \in \mathfrak{W}$, there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$, then $\widehat{v}(\sigma_\lambda - \sigma_{\lambda'}) < \varepsilon$. In particular, for each $x \in X$, $v^x(\sigma_\lambda(x) - \sigma_{\lambda'}(x)) < \varepsilon$, so that $(\sigma_\lambda(x))$ is Cauchy in E_x for each $x \in X$, and hence converges pointwise, say to $\sigma(x) \in E_x$. Suppose that σ (defined pointwise) is unbounded. Then there exist $v \in \mathfrak{W}$ and a sequence $(x_n) \subset X$ such that $v(\sigma(x_n)) > 2n$. On the other hand, there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$, then $\widehat{v}(\sigma_\lambda - \sigma_{\lambda'}) < n$; in particular, $v^{x_n}(\sigma_\lambda(x_n) - \sigma_{\lambda_0}(x_n)) < n$ for all n . Then $\lim_\lambda v^{x_n}(\sigma_\lambda(x_n) - \sigma_{\lambda_0}(x_n)) = v^{x_n}(\sigma(x_n) - \sigma_{\lambda_0}(x_n)) \leq n$ for all n , and so $v^{x_n}(\sigma_{\lambda_0}(x_n)) > n$, a contradiction since $\sigma_{\lambda_0} \in \Gamma_b(\pi)$.

We now claim σ is continuous. Let $x \in X$, and let K be a compact neighborhood of x . Let $w \in \mathfrak{W}$ and $\varepsilon > 0$. Then there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$ then

$$\widehat{\chi_K w}(\sigma_\lambda - \sigma_{\lambda'}) = \sup_{x \in K} w^x(\sigma_\lambda - \sigma_{\lambda'}) = \widehat{w_K}(\sigma_\lambda - \sigma_{\lambda'}) < \varepsilon;$$

i.e. (σ_λ) is u_K -Cauchy on K . Hence by [4, Theorem 5.9], (σ_λ) converges uniformly to σ_K , the restriction of σ to K . But since K is a neighborhood of x , σ_K and thus σ are continuous at x . \square

Proposition 15. *Let X be first countable, and let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4). Suppose that for each $x \in X$, E_x is complete, and let $\Gamma_b(\pi)$ be the space of bounded sections of the bundle $\pi : \mathcal{E} \rightarrow X$ whose topology is determined by \mathcal{F} and \mathfrak{W} . Then $(\Gamma_b(\pi), \beta)$ is complete.*

Proof. Let (σ_λ) be a β -Cauchy sequence in $\Gamma_b(\pi)$. As above, the pointwise limit σ is bounded. We claim σ is continuous. If, say, σ is discontinuous at $x \in X$, then there exist $\varepsilon > 0$, $v \in \mathfrak{W}$, $\tau \in \Gamma_b(\pi)$ with $\tau(x) = \sigma(x)$, an open neighborhood U of x , and a sequence (x_n) of distinct points with $x_n \rightarrow x$ such that $\sigma(x_n) \notin T = T(U, \tau, v, \varepsilon)$. Since (x_n) is eventually in U , this is equivalent to saying (by passing to a subsequence if necessary), that $v^{x_n}(\sigma(x_n) - \tau(x_n)) \geq \varepsilon$ for all n .

Let $a = \chi_B$, the characteristic function of the compact set $B = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Then a is upper semicontinuous and vanishes at infinity. Since (σ_λ) is β -Cauchy, and since $av \in \mathfrak{W}$ (because \mathcal{S}_0 is closed under multiplication) there exists λ_0 such that if $\lambda, \lambda' \geq \lambda_0$, then $\widehat{av}(\sigma_\lambda - \sigma_{\lambda'}) < \varepsilon/2$. Hence, $\sup_{y \in B} \{v^y(\sigma_\lambda(y) - \sigma_{\lambda_0}(y))\} < \varepsilon/2$ whenever $\lambda \geq \lambda_0$. Passing to the limit in λ , we have $\sup_{y \in B} \{v^y(\sigma(y) - \sigma_{\lambda_0}(y))\} \leq \varepsilon/2$, or $v^{x_n}(\sigma(x_n) - \sigma_{\lambda_0}(x_n)) \leq \varepsilon/2$ for all n . This forces $v^{x_n}(\sigma_{\lambda_0}(x_n) - \tau(x_n)) > \varepsilon/2$ for all n . However, $\sigma_{\lambda_0} \in \Gamma_b(\pi)$, and since $x_n \rightarrow x$, we must have $\sigma_{\lambda_0}(x_n) \in T(U, \tau, v, \varepsilon/2)$ for large n . This is a contradiction, since x_n will be in U eventually, but $v^{x_n}(\sigma_{\lambda_0}(x_n) - \tau(x_n)) > \varepsilon/2$ implies that $\sigma_{\lambda_0}(x_n) \notin T$. Hence, σ is continuous. \square

Suppose now that \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), with the following provisos: each E_x is a Banach space, and $\mathfrak{W} = \{\|\cdot\|\}$ is a singleton. Then the bundle $\pi : \mathcal{E} \rightarrow X$ determined by \mathcal{F} and \mathfrak{W} is a bundle of Banach spaces (Banach bundle), and $\Gamma_b(\pi)$ is complete in the sup norm. Moreover, from [9, Corollary 15], if $\pi : \mathcal{E} \rightarrow X$ is a Banach bundle, and if $K \subset X$ is compact, then for $\tau \in \Gamma(\pi_K)$, the section space of the restriction bundle $\pi_K : \mathcal{E}_K \rightarrow X$, there is a norm-preserving extension $\tau' \in \Gamma_b(\pi)$ of τ . Generally speaking, in $\Gamma(\pi)$ for a Banach bundle, the function $x \mapsto \|\sigma(x)\|$ is upper semicontinuous for all $\sigma \in \Gamma(\pi)$; if this function should happen to be continuous for all $\sigma \in \Gamma(\pi)$, call π a continuous bundle.

Recall that a topological space X is a k^* -space if whenever $a : X \rightarrow \mathbb{R}$ is a bounded function such that a_K (the restriction of a to K) is in $C(K)$ for each compact $K \subset X$, then also $a \in C(X)$.

Proposition 16. *Suppose that $\pi : \mathcal{E} \rightarrow X$ is a continuously-normed bundle of Banach spaces, and that $(\Gamma_b(\pi), \beta)$ is complete. Suppose also that*

for each compact $K \subset X$, there exists $\sigma^K \in \Gamma(\pi)$ such that $\|\sigma^K(x)\| > 0$ for all $x \in K$. Then X is a k^* -space.

Proof. Note first that there do exist continuously normed bundles where no (global) section is bounded away from 0; see [8, Example 2], so the assumption is necessary. Let $a : X \rightarrow \mathbb{R}$ be bounded, with $a(x) \geq 0$ for all x , and suppose that for each compact $K \subset X$, $a_K \in C(K)$. For a given compact K , choose $\sigma^K \in \Gamma_b(\pi)$ such that $\|\sigma^K(x)\| > 0$ for all $x \in K$; then $(\sigma^K)_K \in \Gamma(\pi_K)$. By multiplying by an appropriate $b_K \in C(K)$ (which is possible because $x \mapsto \|\sigma^K(x)\|$ is bounded away from 0 on K), we may as well assume that $\|(\sigma^K)_K(x)\| = a(x) = a_K(x) \leq \|a\|$ for each $x \in K$. Let τ^K be a norm-preserving extension of $(\sigma^K)_K$ to all of X ; then $\tau^K \in \Gamma_b(\pi)$, and $x \mapsto \|\tau^K(x)\|$ extends a_K to all of X , preserving the norm.

Order $\mathfrak{K} = \{K \subset X : K \text{ is compact}\}$ by inclusion, and consider the resulting net $\{\tau^K \in \Gamma_b(\pi) : K \in \mathfrak{K}\}$. Let $v \in \mathfrak{B}$, let $\varepsilon > 0$, and choose $K_0 \in \mathfrak{K}$ such that $v(x) < \frac{\varepsilon}{2\|a\| + 1}$ for $x \notin K_0$. Suppose that $K, K' \supset K_0$.

We then have:

$$v(x) \left\| \tau^K(x) - \tau^{K'}(x) \right\| \begin{cases} = 0, & \text{if } x \in K_0 \text{ (}\tau^K \text{ and } \tau^{K'} \text{ both extend } \tau^{K_0}\text{),} \\ < v(x)(\|\tau^K(x)\| + \|\tau^{K'}(x)\|) \\ < v(x) \cdot 2\|a\| < \varepsilon, & \text{if } x \notin K_0. \end{cases}$$

So, $\{\tau^K\}$ is a Cauchy net in $(\Gamma_b(\pi), \beta)$, and hence converges to some $\tau \in \Gamma_b(\pi)$. Necessarily we have $\|\tau(x)\| = a(x)$, and so $a \in C_b(X)$.

Finally, if a is any bounded function on X , we can write $a = a^+ - a^-$ in the usual fashion, and obtain $a \in C(X)$. \square

Corollary 17. *If $(C_b(X), \beta)$ is complete, then X is a k^* -space.*

Corollary 18. *Let $\pi : \mathcal{E} \rightarrow X$ be a continuously normed bundle of commutative Banach algebras E_x with identities e_x , and suppose that the identity selection $e \in \Gamma_b(\pi)$, where $e(x) = e_x$. If $(\Gamma_b(\pi), \beta)$ is complete, then X is a k^* -space.*

Proof. We refer the reader to e.g. [12] or [9] for the somewhat natural definition of a bundle of Banach algebras. Immediately to hand, for each compact $K \subset X$, we have $\|e(x)\| = \|e_x\| \geq 1$ for all $x \in K$, so e is bounded away from 0. \square

Corollary 19. *Let $\pi : \mathcal{E} \rightarrow X$ be the trivial bundle over X with constant fiber the Banach space E . If $(\Gamma_b(X), \beta)$ is complete, then X is a k^* -space.*

Proof. $\Gamma_b(\pi)$ is isometrically isomorphic to $C_b(X, E)$, the space of continuous and bounded E -valued functions on X . \square

Contrast this to the situation with $(C_b(X), \beta)$: this space is complete if and only if X is a k^* -space ([7, Theorem 1]). I.e. completeness of $(C_b(X), \beta)$ depends intrinsically on the topology of X . The inability to characterize completeness of $(\Gamma_b(\pi), \beta)$ in such a fashion would not be surprising, because even in the class of all Banach bundles $\pi : \mathcal{E} \rightarrow X$, the topology of \mathcal{E} , and hence the description of $\Gamma_b(\pi)$, even though it is related to that of X , does not determine the topology of X . For example, if X is compact and infinite, then there are at least two highly non-homeomorphic bundles over X with scalar fibers, namely the trivial bundle $\pi_1 : \mathcal{E}_1 \rightarrow X$, where $\mathcal{E}_1 = X \times \mathbb{R}$ with the product topology, and $\pi_2 : \mathcal{E}_2 \rightarrow X$, where $\mathcal{E}_2 = X \times \mathbb{R}$ with the spiky topology.

We now discuss the beginnings of a notion of compactness within $(\Gamma_b(\pi), \beta)$.

Proposition 20. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and let $\pi : \mathcal{E} \rightarrow X$ be the bundle determined by them. Then a set $B \subset \Gamma_b(\pi)$ is u -bounded if and only if it is β -bounded.*

Proof. This mimics the proof of [10, Theorem 3.4.(iii)]; only one direction needs to be shown. Suppose that $B \subset \Gamma_b(\pi)$ is β -bounded but not u -bounded. Then there exist $w \in \mathfrak{W}$, and a subbasic neighborhood $N(0, \widehat{w}, \varepsilon)$ such that $B \not\subset n^2 \cdot N(0, \widehat{w}, \varepsilon)$ for any n . Thus, we can find a sequence $(\sigma_n) \subset B$ such that $\sigma_n \notin n^2 \cdot N(0, w^x, \varepsilon) \subset E_x$. Hence there is a sequence $(x_n) \subset X$ such that $w^{x_n}(\sigma_n(x_n)) \geq n^2\varepsilon$. Let $a(x_n) = 1/n$ (assuming no repetition on the x 's), and $a(x) = 0$ otherwise. Then a vanishes at infinity. Consider the β -neighborhood $N(0, \widehat{aw}, \varepsilon)$ of 0. Since $(aw)^{x_n}(\sigma_n(x_n)) \geq n\varepsilon$, it follows that $\widehat{aw}(\sigma_n) \geq n\varepsilon$, and so $\sigma_n \notin n \cdot N(0, \widehat{aw}, \varepsilon)$ for any n . Thus, $B \not\subset n \cdot N(0, \widehat{aw}, \varepsilon)$ for any n , which is a contradiction. \square

Proposition 21. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and let $\pi : \mathcal{E} \rightarrow X$ be the bundle determined by them. Let (σ_n) be a net in $\Gamma_b(\pi)$ such that 1) (σ_n) is u -bounded; 2) that for each compact $K \subset X$, $\sigma_n \xrightarrow{u_K} \sigma_K$, where σ_K is the restriction to K of $\sigma \in \Gamma_b(\pi)$. Then $\sigma_n \xrightarrow{\beta} \sigma$.*

Proof. Let $v = aw \in \mathfrak{V}$, and let $\varepsilon > 0$ be given. There exists a compact set $K \subset X$ such that $a(x) < \frac{\varepsilon}{\sup_n \{\widehat{w}(\sigma_n)\} + \widehat{w}(\sigma)}$ if $x \notin K$. We can choose N such that if $n \geq N$, then $\widehat{w}_K(\sigma_n - \sigma) < \frac{\varepsilon}{\|a\|}$. Then for $x \in X$ we have

$$\begin{aligned} (aw)^x(\sigma_n(x) - \sigma(x)) &= a(x)w^x(\sigma_n(x) - \sigma(x)) \\ &\leq \begin{cases} \|a\| \widehat{w}_K(\sigma_n - \sigma), & \text{if } n \geq N \text{ and if } x \in K, \\ \frac{\varepsilon \widehat{w}_K(\sigma_n - \sigma)}{\sup_n \{\widehat{w}(\sigma_n)\} + \widehat{w}(\sigma)} < \varepsilon, & \text{if } n \geq N, x \notin K. \end{cases} \end{aligned}$$

□

The converse is also true.

Proposition 22. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and let $\pi : \mathcal{E} \rightarrow X$ be the bundle determined by them. Suppose that $(\sigma_n) \subset \Gamma_b(\pi)$ is a net such that $\sigma_n \xrightarrow{\beta} \sigma \in \Gamma_b(\pi)$. Then (σ_n) is u -bounded, and $\sigma_n \xrightarrow{u_K} \sigma_K$ for each compact $K \subset X$.*

Proof. Suppose that $\sigma_n \xrightarrow{\beta} \sigma \in \Gamma_b(\pi)$. Then given $v = aw \in \mathfrak{W}$ and $\varepsilon > 0$, there exists N_v such that if $n \geq N_v$, then $\widehat{v}(\sigma_n - \sigma) < \varepsilon$. Thus, $\sup_n \widehat{v}(\sigma_n) < \infty$, and since $v \in \mathfrak{W}$ was arbitrary, $\{\sigma_n\}$ is β -bounded and hence u -bounded.

Now, let $K \subset X$ be compact, and let $\varepsilon > 0$ and $w \in \mathfrak{W}$. Then $\chi_K w \in \mathfrak{W}$, and so for sufficiently large n we have $\widehat{v}(\sigma_n - \sigma) = \sup_{x \in K} w^x(\sigma_n(x) - \sigma(x)) = \widehat{w_K}(\sigma_n - \sigma) < \varepsilon$. But this is what we mean by u_K -convergence. □

Corollary 23. *Let \mathcal{F} and \mathfrak{W} satisfy conditions C1) – C4), and let $\pi : \mathcal{E} \rightarrow X$ be the bundle determined by them. Let (σ_λ) be a net in $\Gamma_b(\pi)$. Then $\sigma_\lambda \xrightarrow{\beta} \sigma \in \Gamma_b(\pi)$ if and only if 1) for each $w \in \mathfrak{W}$ there exists λ_w such that $(\widehat{w}(\sigma_\lambda))_{\lambda \geq \lambda_w}$ is bounded (in \mathbb{R}); and 2) $(\sigma_\lambda)_K \xrightarrow{u_K} \sigma_K$ on each compact $K \subset X$.*

These results are evidently trying to lead toward an Arzelà–Ascoli type result regarding compactness in $(\Gamma_b(\pi), \beta)$: a set $B \subset \Gamma_b(\pi)$ is β -compact (if and only if, one hopes) it is closed and bounded, and it exhibits some sort of property which would reasonably be labelled uniform equicontinuity. But here is where a problem arises. Consider the space $C(X, E)$. Here, there is a natural definition of equicontinuity: a family $B \subset C(X, E)$ is equicontinuous at x if given any neighborhood \mathcal{N} of 0 in E , there exists a neighborhood U of x such that for all $\sigma \in B$, $\sigma(x) - \sigma(y) \in \mathcal{N}$ whenever $x, y \in U$. Because $\sigma(x)$ and $\sigma(y)$ both are in E , the subtraction makes sense. Yet a straightforward attempt to apply this to our more general situation fails. We could, for example, try: $B \subset \Gamma_b(\pi)$ is equicontinuous at x provided that for each $w \in \mathfrak{W}$ and $\varepsilon > 0$, there exists a neighborhood U of x such that if $x, y \in U$, then — what? Since $\sigma(x)$ and $\sigma(y)$ are in different spaces (given the context), $\sigma(x) - \sigma(y)$ is undefined. And to have both $\sigma(x), \sigma(y) \in T(U, \sigma, \widehat{v}, \varepsilon)$ says nothing more than σ is continuous. It is thus unclear how to proceed.

Acknowledgement. The authors wish to thank the anonymous referee for his/her careful reading and editorial suggestions.

References

- [1] J. Arhippainen and J. Kauppi, *Generalization of the topological algebra $(C_b(X), \beta)$* , *Studia Math.* **191** (2009), 247–262.

- [2] H. Arizmendi-Peimbert, A. Carillo-Hoyo, and A. García, *A spectral synthesis property for $(C_b(X), \beta)$* , Comment. Math. **48** (2008), 121–127.
- [3] R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.
- [4] G. Gierz, *Bundles of Topological Vector Spaces and Their Duality*, Lect. Notes in Math. **955**, Springer-Verlag, Berlin, 1982.
- [5] R. Giles, *A generalization of the strict topology*, Trans. Amer. Math. Soc. **161** (1971), 467–474.
- [6] A. E. Gutman and A. V. Koptev, *On the notion of the dual of a Banach bundle*, Siberian Adv. Math. **9** (1999), 46–98.
- [7] J. Hoffman-Jørgensen, *A generalization of the strict topology*, Math. Scand. **30** (1972), 313–323.
- [8] T. Hõim and D. A. Robbins, *Isomorphisms into section spaces of Banach bundles*, Quaest. Math. **30** (2007), 97–113.
- [9] T. Hõim and D. A. Robbins, *Spectral synthesis and other results in some topological algebras of vector-valued functions*, Quaest. Math. (to appear)
- [10] L. A. Khan, *The strict topology on a space of vector-valued functions*, Proc. Edinburgh Math. Soc. **22** (1979), 35–41.
- [11] J. W. Kitchen and D. A. Robbins, *Gelfand Representation of Banach Modules*, Dissertationes Math. (Rozprawy Mat.) **203** (1982).
- [12] J. W. Kitchen and D. A. Robbins, *Sectional representation of Banach modules*, Pacific J. Math. **109** (1983), 135–156.
- [13] J. W. Kitchen and D. A. Robbins, *Two notions of spectral synthesis for Banach modules*, Tamkang J. Math. **16** (1985), 59–65.
- [14] J. W. Kitchen and D. A. Robbins, *Maximal ideals in algebras of vector-valued functions*, Internat. J. Math. Math. Sci. **19** (1996), 549–554.
- [15] W. Summers, *Weighted approximation for modules of continuous functions*, Bull. Amer. Mat. Soc. **79** (1973), 386–388.
- [16] C. Todd, *Stone–Weierstrass theorems for the strict topology*, Proc. Amer. Math. Soc. **16** (1965), 654–659.

DEPARTMENT OF MATHEMATICS, WILKES HONORS COLLEGE, FLORIDA ATLANTIC UNIVERSITY, JUPITER, FL 33458, USA

E-mail address: `thoim@fau.edu`

DEPARTMENT OF MATHEMATICS, TRINITY COLLEGE, HARTFORD, CT 06106, USA

E-mail address: `david.robbins@trincoll.edu`