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ON THE STRUCTURE OF GRAPHS WITH NON-SURJECTIVE $L(2,1)$-LABELINGS*

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Abstract. For a graph G, an $L(2,1)$-labeling of G with span k is a mapping $L \to \{0,1,2,\ldots,k\}$ such that adjacent vertices are assigned integers which differ by at least 2, vertices at distance two are assigned integers which differ by at least 1, and the image of L includes 0 and k. The minimum span over all $L(2,1)$-labelings of G is denoted $\lambda(G)$, and each $L(2,1)$-labeling with span $\lambda(G)$ is called a λ-labeling. For $h \in \{1,\ldots,k-1\}$, h is a hole of L if and only if h is not in the image of L. The minimum number of holes over all λ-labelings is denoted $\rho(G)$, and the minimum k for which there exists a surjective $L(2,1)$-labeling onto $\{0,1,\ldots,k\}$ is denoted $\mu(G)$. This paper extends the work of Fishburn and Roberts on ρ and μ through the investigation of an equivalence relation on the set of λ-labelings with ρ holes. In particular, we establish that $\rho \leq \Delta$. We analyze the structure of those graphs for which $\rho \in \{\Delta-1,\Delta\}$, and we show that $\mu = \lambda + 1$ whenever λ is less than the order of the graph. Finally, we give constructions of connected graphs with $\rho = \Delta$ and order $t(\Delta+1)$, $1 \leq t \leq \Delta$.

Key words. $L(2,1)$-labeling, λ-labeling, hole index, dominating vertex set

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1. Introduction. The $L(2,1)$-labeling problem is a vertex-labeling analog of Hale's channel assignment problem [14] which seeks to minimize the range of frequencies used while at the same time ensuring that transmitters which are sufficiently close together are assigned transmission frequencies which differ by no less than a prescribed amount.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For fixed positive integer k, an $L(2,1)$-*labeling of* G with *span* k is a mapping L from $V(G)$ into $\{0,1,2,\ldots,k\}$ such that any two vertices which are adjacent are assigned integers which differ by at least 2, any two vertices which are distance two apart are assigned integers which differ by at least 1, and the integers 0 and k are each assigned to at least one vertex. We denote the span k of L by $s(L)$, and for each vertex $v \in V(G)$, we refer to $L(v)$ as the label of v assigned by L. The minimum span among all $L(2,1)$-labelings of G is called the λ-*number of* G, denoted $\lambda(G)$. Any $L(2,1)$-labeling which achieves a span of $\lambda(G)$ is called a λ-*labeling of* G.

For an $L(2,1)$-labeling L of G and for integer h such that $0 < h < s(L)$, h *is a hole of* L if and only if h is not assigned by L to any vertex v in $V(G)$. The minimum number of holes over all λ-labelings of G is called the *hole index of* G, and is denoted $\rho(G)$. If there exists a λ-labeling L of G with no holes, then L is called a *no-hole* λ-*labeling of* G and G is said to be λ-*full-colorable*. Alternatively, G is λ-full-colorable if and only if there exists a surjective λ-labeling of G. If there exists an $L(2,1)$-labeling of G (not necessarily a λ-labeling) with no holes, then the minimum span over all such $L(2,1)$-labelings of G is denoted $\mu(G)$. Clearly, $\mu(G) \geq \lambda(G)$, and $\mu(G) = \lambda(G)$ if and only if $\rho(G) = 0$.

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The $L(2,1)$-labeling was introduced by Griggs and Yeh [13] as an extension of T-colorings (see [16]). There, they considered the λ-numbers of graphs in various classes such as trees, cycles, and paths, and they investigated the relationship between $\lambda(G)$ and other graph invariants such as $\Delta(G)$ and $\chi(G)$. Since then, many other authors have extended these lines of study, exploring the λ-numbers of the n-cube [19], chordal graphs [17], various products of graphs [10, 11, 15], as well as exploring the relationship between $\lambda(G)$ and other invariants such as the size of G [9] and the path covering number of G^c (the complement of G) [12]. Generalizations of $L(2,1)$-labelings have also been considered; see [2, 4, 8, 10, 11, 18].

Recently, attention has turned to the study of graphs G for which $\rho(G) = 0$. Fishburn and Roberts [6, 7] in particular have shown that $\rho(G) = 0$ if $|V(G)| = \lambda(G) + 1$, and that $\rho(G) = 0$ if G is any tree distinct from the claw $K_{1,n}$. They have constructed a number of graphs G with $\rho(G) > 0$, and, in the event that $\rho(G) > 0$, they have shown that $\lambda(G) + \rho(G)$ is an upper bound for $\mu(G)$ if $\mu(G)$ exists.

In this paper, we continue the study of $\rho(G)$ with emphasis on $\rho(G) > 0$. Section 2 provides notation, definitions, and an equivalence class on the set of λ-labelings of G with $\rho(G)$ holes which will facilitate our discussion. We consider the relationship between $\rho(G)$ and $\Delta(G)$ (section 3) and the relationships among $\rho(G), \mu(G)$, and $\lambda(G)$ (section 4). In section 5, we explore the structure of graphs with the property $\rho(G) = \Delta(G)$.

2. Definitions and preliminary results. The sum $G_1 + G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ with $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$.

Let L be an $L(2,1)$-labeling of G. Then $M_i(G,L) = \{v \in V(G) | L(v) = i\}$ and $m_i(G,L) = |M_i(G,L)|$.

Let L be a λ-labeling of G. Suppose $0 < h_1 < h_2 < h_3 < \cdots < h_w < \lambda(G)$ are the holes of L. Then for k, $1 \le k \le w-1$, the set of integers strictly between h_k and h_{k+1} shall be called *island k of L*, denoted $I_k(L)$. Similarly, *island* 0 *of* L, denoted $I_0(L)$, and *island w of L*, denoted $I_w(L)$, shall, respectively, mean $\{0, 1, 2, \ldots, h_1 - 1\}$ and $\{h_w + 1, h_w + 2, \ldots, \lambda(G)\}$. For $0 \le k \le w$, the smallest element of $I_k(L)$ shall be called the *left coast of* $I_k(L)$ (denoted $\mathrm{lc}(I_k(L))$) and the largest element of $I_k(L)$ shall be called the *right coast of* $I_k(L)$ (denoted $\mathrm{rc}(I_k(L))$). Integers which are the left coast or right coast of some island will be called *coastal labels.* The *interior of* $I_k(L)$, denoted $\mathrm{int}(I_k(L))$, shall mean $I_k(L) - \{\mathrm{lc}(I_k(L)), \mathrm{rc}(I_k(L))\}$. The set of coastal labels in island $I_k(L)$ will be denoted $C(I_k(L))$. In the case of the equivalent conditions $|C(I_k(L))| = 1, |I_k(L)| = 1$, and $\mathrm{lc}(I_k(L)) = \mathrm{rc}(I_k(L))$, we shall refer to $I_k(L)$ as an *atoll.*

For any island $I_j(L) = \{x, x+1, \ldots, x+z\}$, we let $Z_j(L)$ denote the sequence of sets of vertices $(M_x(G,L), M_{x+1}(G,L), \ldots, M_{x+z}(G,L))$. We also define $Z(L)$ to be the sequence $(Z_0(L), Z_1(L), Z_2(L), \ldots, Z_w(L))$.

For any graph G, let $\Lambda_\rho(G)$ be the collection of all λ-labelings of G with $\rho(G)$ holes. Also, let $\mathcal{L}(G,t)$ be the collection of $L(2,1)$-labelings of G with span t. It is clear that if $L \in \mathcal{L}(G,t)$, then the labeling $L' = t - L$ is also in $\mathcal{L}(G,t)$. We therefore observe that $v \in M_i(G,L)$ if and only if $v \in M_{t-i}(G,L')$.

We next define and illustrate two classes of vertex labelings of G, elements of which follow from a given labeling $L \in \Lambda_\rho(G)$.

For any $L \in \Lambda_\rho(G)$ and any island $I_j(L)$, define

$$\phi_j(L)(v) = \begin{cases} L(v) & \text{if } L(v) \notin I_j(L), \\ \mathrm{rc}(I_j(L)) - i & \text{if } L(v) = \mathrm{lc}(I_j(L)) + i \in I_j(L). \end{cases}$$

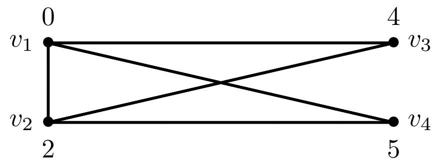

FIG. 2.1. *$L(2,1)$-labeling of $K_{1,1,2}$.*

We call this labeling of the vertices of G an *intra-island relabeling at L,* and note that $\phi_j(L)$ is easily seen to be an element of $\Lambda_\rho(G)$ with holes identical to the holes of L. It therefore follows that the composition of any number of intra-island relabelings at L is an element of $\Lambda_\rho(G)$. We observe that the components of $Z_j(\phi_j(L))$ are the components of $Z_j(L)$ in opposite order. (For $k \neq j$, $Z_k(\phi_j(L)) = Z_k(L)$.) We also observe that the relation Φ on $\Lambda_\rho(G)$, given by $(L_1, L_2) \in \Phi$ if and only if L_2 is a finite composition of intra-island relabelings at L_1, is an equivalence relation. Moreover, the cardinality of the equivalence class containing L is $2^{\rho(G)+1-a}$, where a is the number of atolls of L.

For any $L \in \Lambda_\rho(G)$ and for a fixed j, $0 \leq j \leq \rho(G) - 1$, define

$$\psi_j(L)(v) = \begin{cases} L(v) & \text{if } L(v) \notin I_j(L) \bigcup I_{j+1}(L) \\ L(v) - (\text{lc}(I_{j+1}(L)) - \text{lc}(I_j(L))) & \text{if } L(v) \in I_{j+1}(L) \\ L(v) + \text{rc}(I_{j+1}(L)) - \text{lc}(I_{j+1}(L)) + 2 & \text{if } L(v) \in I_j(L). \end{cases}$$

We call this labeling of G an *inter-island relabeling at L,* and note that $\psi_j(L)$ is an element of $\Lambda_\rho(G)$ with the following properties:

1. $\psi_j(L)$ has a hole at $\text{lc}(I_j(L)) + \text{rc}(I_{j+1}(L)) - \text{lc}(I_{j+1}(L)) + 1$;
2. $Z_{j+1}(\psi_j(L)) = Z_j(L)$;
3. $Z_j(\psi_j(L)) = Z_{j+1}(L)$.

We also note that since $\psi_j(L) \in \Lambda_\rho(G)$, it follows that the composition of any finite number of inter-island relabelings at L is an element of $\Lambda_\rho(G)$ as well.

Example 2.1. Consider the graph $G = K_{1,1,2}$ along with an $L(2,1)$-labeling L as given in Figure 2.1.

Since it is easily seen that $\lambda(G) = 5$ and $\rho(G) = 2$, then $L \in \Lambda_\rho(G)$ with islands $I_0(L) = \{0\}$, $I_1(L) = \{2\}$ and $I_2(L) = \{4, 5\}$. Thus,

$$\psi_1(L)(v) = \begin{cases} 0 & \text{if } v = v_1, \\ 5 & \text{if } v = v_2, \\ 2 & \text{if } v = v_3, \\ 3 & \text{if } v = v_4, \end{cases}$$

and the islands of $\psi_1(L)$ are $\{0\}, \{2, 3\}$, and $\{5\}$.

Additionally,

$$\phi_2(L)(v) = \begin{cases} 0 & \text{if } v = v_1, \\ 2 & \text{if } v = v_2, \\ 5 & \text{if } v = v_3, \\ 4 & \text{if } v = v_4. \end{cases}$$

We next note that for any finite composition $\psi(L)$ of inter-island relabelings at L, there exists a permutation θ of $\{0, 1, 2, \ldots, \rho(G)\}$ such that

$$Z(\psi(L)) = (Z_{\theta^{-1}(0)}(L), Z_{\theta^{-1}(1)}(L), \ldots, Z_{\theta^{-1}(\rho(G))}(L)).$$

And, conversely, for every permutation θ of $\{0,1,2,\ldots,\rho(G)\}$, there exists a finite composition $\psi(L)$ of inter-island relabelings at L such that $Z(\psi(L)) = (Z_{\theta^{-1}(0)}(L), Z_{\theta^{-1}(1)}(L),\ldots,Z_{\theta^{-1}(\rho(G))}(L))$. It follows that for any $L \in \Lambda_\rho(G)$ with islands $I_0(L), I_1(L),\ldots,I_{\rho(G)}(L)$, there is a composition ψ of inter-island relabelings at L with islands $I_0(\psi(L)), I_1(\psi(L)),\ldots,I_{\rho(G)}(\psi(L))$ such that $|I_0(\psi(L))| \le |I_1(\psi(L))| \le \cdots \le |I_{\rho(G)}(\psi(L))|$. Thus, without losing the generality of G, we shall assume $|I_0(L)| \le |I_1(L)| \le \cdots \le |I_{\rho(G)}(L)|$ when convenient.

Example 2.2. Let G be a graph with $\rho(G) = 2$ and let $L \in \Lambda_\rho(G)$. Let $\psi(L) = \psi_0 \circ \psi_1(L)$. Then

$$Z(L) = (Z_0(L), Z_1(L), Z_2(L))$$

and

$$Z(\psi(L)) = (Z_2(L), Z_0(L), Z_1(L)).$$

It is easy to see that the relation Ψ on $\Lambda_\rho(G)$, given by $(L_1, L_2) \in \Psi$ if and only if $L_2 = \psi(L_1)$ for some finite composition ψ of inter-island relabelings at L_1, is an equivalence relation. Moreover, the cardinality of each equivalence class under Ψ is $(\rho(G)+1)!$.

Finally, we observe that the relation Ω on $\Lambda_\rho(G)$, given by $(L_1, L_2) \in \Omega$ if and only if $L_2 = \omega(L_1)$ for some finite composition ω of inter- and/or intra-island relabelings at L_1, is an equivalence relation, and that there are $(\rho(G)+1)!2^{\rho(G)+1-a}$ members in each equivalence class containing L_1, where a is the number of atolls of L_1.

Example 2.3. If $G = K_{2,3}$, then $\lambda(G) = 5$ and $\rho(G) = 1$. Furthermore, every λ-labeling of G is in $\Lambda_\rho(G)$, each such labeling induces 2 islands (one with cardinality two and one with cardinality three), and $|\Lambda_\rho(G)| = 24$. Finally, for $L \in \Lambda_\rho(G)$, $|[L]_\Phi| = 4$, $|[L]_\Psi| = 2$, and $|[L]_\Omega| = 8$.

Example 2.4. If $G = K_2 + K_4$, then $\lambda(G) = 6$ and $\rho(G) = 1$. The graph G has 720 different λ-labelings, of which 144 are in $\Lambda_\rho(G)$. Among the islands in $\Lambda_\rho(G)$, 48 induce 2 islands of cardinality 3 each, and the other 96 labelings induce 2 islands with cardinalities 1 and 5. We are not aware of the existence of a connected graph having $\rho(G) \ge 1$ which has two labelings which induce islands having different cardinalities as illustrated in the analysis of the disconnected graph $K_2 + K_4$.

We close this section with a definition and related theorem which will prove useful in section 4.

Let H be a graph. Then a *path covering of* H is a set of vertex-disjoint paths in H which cover $V(H)$. The path-covering number of H, denoted $c(H)$, is the minimum cardinality over all path coverings of H.

THEOREM 2.5 ([12]). *Suppose G is a graph with $|V(G)| = n$. Then*

1. $\lambda(G) = n + c(G^c) - 2$ *if* $c(G^c) \ge 2$
2. $\lambda(G) \le n - 1$ *if* $c(G^c) = 1$.

3. Relating $\rho(G)$ and $\Delta(G)$. In this section, we make use of configurations of islands to explore the relationship between $\rho(G)$ and $\Delta(G)$.

LEMMA 3.1. *Let G be a graph with $\rho(G) \ge 1$, let $L \in \Lambda_\rho(G)$ and let $0 \le i < j \le \rho(G)$. Suppose $x \in \{lc(I_i(L)), rc(I_i(L))\}$ and $y \in \{lc(I_j(L)), rc(I_j(L))\}$. Then*

1. *for each $v \in M_x(G,L)$, there exists a unique vertex $w \in M_y(G,L)$ such that w and v are adjacent, and*
2. $m_x(G,L) = m_y(G,L)$.

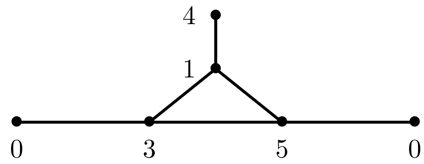

FIG. 3.1. *Graph G with $\rho(G) = 0$.*

Proof. Through some finite composition ω of inter- and/or intra-island relabelings at L, we may construct an element $\omega(L)$ of $\Lambda_\rho(G)$ such that for some α, α is a hole of $\omega(L)$, $M_x(G, L) = M_{\alpha-1}(G, \omega(L))$, and $M_y(G, L) = M_{\alpha+1}(G, \omega(L))$.

Proof of (1). Select $v \in M_{\alpha-1}(G, \omega(L))$, and suppose to the contrary that for every vertex $w \in M_{\alpha+1}(G, \omega(L))$, $\{v, w\} \notin E(G)$. Select vertex $w' \in M_{\alpha+1}(G, \omega(L))$. If $|M_{\alpha+1}(G, \omega(L))| \geq 2$, we produce an $L(2, 1)$-labeling L' of G with $\rho(G) - 1$ holes

$$L'(u) = \begin{cases} \omega(L)(u) & \text{if } u \neq w', \\ \omega(L)(u) - 1 & \text{if } u = w', \end{cases}$$

contradicting that $\omega(L)$ is a λ-labeling with the minimum number of holes. On the other hand, if $|M_{\alpha+1}(G, \omega(L))| = 1$, then we produce an $L(2, 1)$-labeling L' of G with span $\lambda(G) - 1$,

$$L'(u) = \begin{cases} \omega(L)(u) & \text{if } \omega(L)(u) \leq \alpha - 1, \\ \omega(L)(u) - 1 & \text{otherwise}, \end{cases}$$

contradicting that $\omega(L)$ is a λ-labeling. Thus, for each $v \in M_x(G, L)$, there exists vertex $w \in M_y(G, L)$ such that w and v are adjacent. Uniqueness of w follows from the distance 2 condition.

Proof of (2) follows immediately from (1). □

Example 3.2. Consider the graph G and $L(2, 1)$-labeling L of G given in Figure 3.1. It is easily verified that L is a λ-labeling of G with one hole at 2; hence $\rho(G) \leq 1$. Since $1 = m_5(G, L) \neq m_0(G, L) = 2$, Lemma 3.1 implies that $\rho(G) < 1$. Hence, there must exist a λ-labeling of G with $\rho(G) = 0$.

When there is no chance of confusion, we may hereafter suppress the functional dependence of the various island notations on L. Likewise, we may suppress the functional dependence of the notations $M_i(G, L)$ and $m_i(G, L)$ on G and L.

LEMMA 3.3. *Let G be a graph with $\rho(G) \geq 1$ and let $L \in \Lambda_\rho(G)$. Then $\Delta(G) \geq \sum_{j=1}^{\rho(G)} |C(I_j)| \geq \rho(G)$.*

Proof. Let v be a vertex with label $\mathrm{rc}(I_0)$ under L. Then from Lemma 3.1, it follows that for $1 \leq j \leq \rho(G)$ and for y in $\{\mathrm{lc}(I_j), \mathrm{rc}(I_j)\}$, v is adjacent to some vertex in M_y. Thus, $\Delta(G) \geq d(v) \geq \sum_{j=1}^{\rho(G)} |C(I_j)| \geq \sum_{j=1}^{\rho(G)} 1 = \rho(G)$. □

Recalling that L exists in $\Lambda_\rho(G)$ such that $|I_0| \leq |I_1| \leq |I_2| \leq \cdots \leq |I_{\rho(G)}|$, we note that the greatest lower bound for $\Delta(G)$ afforded by $\sum_{j=1}^{\rho(G)} |C(I_j)|$ occurs at such L. We also note that the following result is an immediate consequence of Lemma 3.3.

THEOREM 3.4. *For any graph G, $\rho(G) \leq \Delta(G)$.*

For the remainder of this section, we shall consider the structures of graphs associated with $\rho(G) = \Delta(G)$ and $\rho(G) = \Delta(G) - 1$, with particular attention paid to Δ-regular graphs.

THEOREM 3.5. *Let G be a graph with $\rho(G) = \Delta(G)$ and let $L \in \Lambda_\rho(G)$. Then*

1. *every island of L is an atoll; particularly, $I_j = \{2j\}$ for $0 \leq j \leq \Delta(G)$.*

2. *$\lambda(G) = 2\Delta(G)$.*
3. *G is Δ-regular and $|V(G)| \equiv 0$ mod $(\Delta(G)+1)$.*
4. *For every j, $0 \le j \le \Delta(G)$, M_{2j} is a dominating set of vertices in G.*

Proof. With no loss of generality, we assume $|I_0| \le |I_1| \le |I_2| \le \cdots \le |I_{\Delta(G)}|$.

Proof of (1). By the monotonicity of the cardinality of the islands, it suffices to show that $|I_{\Delta(G)}| = 1$. Suppose to the contrary that $|I_{\Delta(G)}| \ge 2$. Then $\sum_{i=1}^{\Delta(G)} |C(I_i)| \ge 2 + \sum_{i=1}^{\Delta(G)-1} |C(I_i)| \ge 2 + \sum_{i=1}^{\Delta(G)-1} 1 \ge \Delta(G) + 1$, contradicting Lemma 3.3. Since each island is thus an atoll and no two holes are consecutive [see Lemma 2.2 in [12]], then $I_j = \{2j\}$ for $0 \le j \le \Delta(G)$.

Proof of (2). From (1), $\mathrm{rc}(I_{\Delta(G)}) = 2\Delta(G)$. But $\lambda(G) = \mathrm{rc}(I_{\rho(G)}) = \mathrm{rc}(I_{\Delta(G)})$, since $\Delta(G) = \rho(G)$.

Proof of (3). Since each vertex of G is assigned a coastal label under L, the result follows from Lemma 3.1.

Proof of (4). For each fixed j, $0 \le j \le \Delta(G)$, and each $i \ne j$, $0 \le i \le \Delta(G)$, each vertex in M_{2i} is adjacent to some vertex in M_{2j} by Lemma 3.1. □

We note that in the next section, additional consideration will be given to the structure of graphs in the case $\Delta(G) = \rho(G)$.

THEOREM 3.6. *Let G be a graph with $\Delta(G) \ge 1$ and $\rho(G) = \Delta(G) - 1$. Then $2\Delta(G) - 1 \le \lambda(G) \le 2\Delta(G)$. Furthermore, if $\lambda(G) = 2\Delta(G)$, then*

1. *If $\Delta(G) = 1$, then $G = mK_2 + nK_1$ where $m, n \ge 1$.*
2. *If $\Delta(G) = 2$, then $G = nC_4$ or $H + K_1$ where H is a graph with $\rho(H) = \Delta(G)$.*
3. *If $\Delta(G) \ge 3$, then $G = H + K_1$ where H is a graph with $\rho(H) = \Delta(G)$.*

Proof. To show that $2\Delta(G) - 1 \le \lambda(G)$, we note that since $K_{1,\Delta(G)}$ is a subgraph of G, every $L(2,1)$-labeling L uses at least $\Delta(G) + 1$ distinct labels. Since $\rho(G) = \Delta(G) - 1$, it follows that $s(L) \ge 2\Delta(G) - 1$.

We next show that $\lambda(G) \le 2\Delta(G)$. For any graph G, if $\Delta(G) = 1$ (resp. 2), then $\lambda(G) = 2$ (resp. ≤ 4). In the case $\Delta(G) \ge 3$, suppose to the contrary that $\lambda(G) \ge 2\Delta(G) + 1$, and let $L \in \Lambda_\rho(G)$ with $|I_0(L)| \le |I_1(L)| \le |I_2(L)| \le \cdots \le |I_{\Delta(G)-1}(L)|$. We observe that $|I_{\Delta(G)-2}(L)| = 1$; otherwise, $\sum_{i=1}^{\Delta(G)-1} |C(I_i(L))| \ge 4 + \sum_{i=1}^{\Delta(G)-3} |C(I_i(L))| \ge 4 + (\Delta(G) - 3) = \Delta(G) + 1$, contradicting Lemma 3.3. Since it follows that $I_j(L) = \{2j\}$ for $0 \le j \le \Delta(G) - 2$, then $I_{\Delta(G)-1}(L) = \{2\Delta(G) - 2, 2\Delta(G) - 1, \ldots, \lambda(G)\}$. But $\lambda(G) \ge 2\Delta(G) + 1$, implying that $|I_{\Delta(G)-1}(L)| \ge 4$ and hence $|C(I_{\Delta(G)-1}(L))| = 2$. Therefore, by the arbitrariness of L, every element of $\Lambda_\rho(G)$ induces $\Delta(G)$ islands, exactly $\Delta(G) - 1$ of which are atolls.

By Lemma 3.1, each vertex in $M_0(G, L)$ has degree $\Delta(G)$ and is thus adjacent only to vertices with labels in $\bigcup_{i=1}^{\Delta(G)-1} C(I_i(L))$. This implies that no vertex with label 0 is adjacent to a vertex with label $2\Delta(G) - 1$. It similarly follows that no vertex with label 2 is adjacent to any vertex with label $2\Delta(G) - 1$. Therefore, given fixed $v_0 \in M_{2\Delta(G)-1}(G, L)$, we may produce a new λ-labeling L' of G as follows:

$$L'(v) = \begin{cases} L(v) & \text{if } v \ne v_0, \\ 1 & \text{if } v = v_0. \end{cases}$$

If $m_{2\Delta(G)-1}(G, L) \ge 2$, then L' has $\Delta(G) - 2 < \rho(G)$ holes, a contradiction of the minimality of $\rho(G)$. If $m_{2\Delta(G)-1}(G, L) = 1$, then L' is in $\Lambda_\rho(G)$ and induces $\Delta(G)$ islands of which exactly $\Delta(G) - 2$ are atolls. But this contradicts the earlier observation that every element of $\Lambda_\rho(G)$ induces $\Delta(G)$ islands, exactly $\Delta(G) - 1$ of which are atolls. These contradictions imply that $\lambda(G) \le 2\Delta(G)$.

We now turn to parts (1), (2), and (3). Suppose $\lambda(G) = 2\Delta(G)$, with $\rho(G) = 2\Delta(G) - 1$.

Proof of (1). Obvious.

Proof of (2). If $\Delta(G) = 2$, then $\lambda(G) = 4$ and $\rho(G) = 1$. If $L \in \Lambda_\rho(G)$, then L induces the following islands:

$I_0 = \{0\}$, $I_1 = \{2,3,4\}$, or

$I_0 = \{0,1,2\}$, $I_1 = \{4\}$, or

$I_0 = \{0,1\}$, $I_1\{3,4\}$.

In the first of these cases, every vertex in M_0 has degree 2, and, by Lemma 3.1, is adjacent to some vertex in M_2 and some vertex in M_4. Thus, no vertex in M_3 is adjacent to a vertex in M_0. Moreover, since no vertex in M_3 can be adjacent to a vertex in M_2 or M_4, then each vertex in M_3 is isolated. Now fix $v \in M_3$. If $m_3 \geq 2$, then we can produce a new λ-labeling L' of G with no holes by relabeling v with 1, contradicting the minimality of $\rho(G)$. Therefore $m_3 = 1$, whence $G = H + K_1$, where $H = (V(G) - \{v\}, E(G))$ is a graph with $\rho(H) = \Delta(G) = 2$. A similar argument can be applied to the case $I_0 = \{0,1,2\}$, $I_1 = \{4\}$.

If $I_0 = \{0,1\}$ and $I_1 = \{3,4\}$, then

1. every vertex in M_0 is adjacent to some vertex in M_3 and some vertex in M_4;
2. every vertex in M_1 is adjacent to some vertex in M_3 and some vertex in M_4;
3. every vertex in M_3 is adjacent to some vertex in M_0 and some vertex in M_1; and
4. every vertex in M_4 is adjacent to some vertex in M_0 and some vertex in M_1.

Thus, G is a 2-regular graph and hence is a sum of cycles. Furthermore, since L has a hole at two, each cycle of G has length $4k$, $k \geq 1$. However, for any $k \geq 2$, it can be easily shown that a cycle of length $4k$ has a λ-labeling with no holes. Thus $k = 1$.

Proof of (3). Suppose $\Delta(G) \geq 3$, $\rho(G) = \Delta(G) - 1$, and $\lambda(G) = 2\Delta(G)$. Let $L \in \Lambda_\rho(G)$ with $|I_0| \leq |I_1| \leq |I_2| \leq \cdots \leq |I_{\Delta(G)-1}|$. Since $\lambda(G) = 2\Delta(G)$ and $\mathrm{lc}(I_{\Delta(G)-2}) \geq 2\Delta(G) - 4$, either $|I_{\Delta(G)-2}| = |I_{\Delta(G)-1}| = 2$ or $|I_{\Delta(G)-2}| = 1$ and $|I_{\Delta(G)-1}| = 3$. In the former case, each vertex in M_0 has degree $\Delta(G) + 1$ by Lemma 3.1, a contradiction. In the latter case, I_j is an atoll for $0 \leq j \leq \Delta(G) - 2$, and $I_{\Delta(G)-1} = \{2\Delta(G) - 2, 2\Delta(G) - 1, 2\Delta(G)\}$. Therefore, by arguments identical to those given for the first case of (2), $M_{2\Delta(G)-1}$ contains exactly one vertex v, and that vertex is isolated. Thus $G = H + K_1$ where $H = (V(G) - \{v\}, E(G))$ is a graph with $\rho(H) = \Delta(G)$. □

Theorem 3.7. *For arbitrary $k \geq 1$, there is no k-regular graph G with $\rho(G) = k - 1$ except for $k = 2$ and $G = nC_4$, $n \geq 1$.*

Proof. Suppose $k \geq 3$ and let G be k-regular with $\rho(G) = k - 1$. By Theorem 3.6(3), $\lambda(G) = 2k - 1$ since G has no isolated vertex. Let $L \in \Lambda_\rho(G)$ with $|I_0| \leq |I_1| \leq |I_2| \leq \cdots \leq |I_{k-1}|$. Then $I_j = \{2j\}$ for $0 \leq j \leq k-2$ and $I_{k-1} = \{2k-2, 2k-1\}$. Let $v \in M_{2k-2}$. Then v can be adjacent only to vertices with labels in I_j, $0 \leq j \leq k-2$, implying $d(v) = k - 1$, a contradiction to the k-regularity of G. The cases $k = 1, 2$ follow from inspection. □

Corollary 3.8. *Let G be a graph with $\delta(G) \geq 1$. If $\delta(G) \leq \Delta(G) - 2$, then $\rho(G) \leq \Delta(G) - 2$.*

Proof. By Theorem 3.4, $\rho(G) \leq \Delta(G)$. If $\rho(G) = \Delta(G)$, then by Theorem 3.5, G is $\Delta(G)$-regular, and $\delta(G) = \Delta(G)$. So, suppose $\rho(G) = \Delta(G) - 1$. Then by Theorem 3.6, $\lambda(G) \leq 2\Delta(G)$. If $\lambda(G) = 2\Delta(G)$, then by Theorem 3.6 and the assumption $\delta(G) \geq 1$, it follows that $\Delta(G) = 2$, implying the contradiction $\delta(G) - 2 \leq 0$. Therefore $\lambda(G) = 2\Delta(G) - 1$. Arguing as above, let $L \in \Lambda_\rho(G)$ with $|I_0| \leq |I_1| \leq |I_2| \leq \cdots \leq |I_{\Delta(G)-1}|$. Then every island under L is necessarily an atoll except $I_{\Delta(G)-1} = \{2\Delta(G) - 2, 2\Delta(G) - 1\}$. So, for $v \in M_{2\Delta(G)-2}$, $d(v) = \Delta(G) - 1$ by Lemma 3.1, a contradiction to the assumption $\delta(G) \leq \Delta(G) - 2$. □

THEOREM 3.9. *Let G be k-regular and let $L \in \Lambda_\rho(G)$ with $|I_0| \leq |I_1| \leq \cdots \leq |I_{\rho(G)}|$. Then*

1. *If $|I_{\rho(G)}| = 1$, then $\rho(G) = k$ and $\lambda(G) = 2k$.*
2. *If $|I_{\rho(G)}| = 2$, then $\rho(G) \geq 1$, $|I_j| = 2$ for all $0 \leq j \leq \rho(G)$, $k = 2\rho(G)$, and $\lambda(G) = 3\rho(G) + 1 = \frac{3}{2}k + 1$.*
3. *If $|I_{\rho(G)}| \geq 3$, then $k \geq 2$, $\rho(G) \leq k - 2$, and $\lambda(G) \geq k + 2 + \rho(G)$.*

Proof. (1) There are $\rho(G)+1$ islands of L, each of which is an atoll since $|I_{\rho(G)}| = 1$. Thus, by Lemma 3.1, $k = \rho(G)$, from which it follows from Theorem 3.5 that $\lambda(G) = 2k$.

Proof of (2). If $\rho(G) = 0$, then $I_0 = \{0, 1\}$, implying the contradiction that $\lambda(G) = 1$. So $\rho(G) \geq 1$. We now show that $|I_j| = 2$ for all $0 \leq j \leq \rho(G)$ by showing that $|I_0| = 2$.

We observe that each island under L contains only coastal labels since $|I_{\rho(G)}| = 2$. Let w be a vertex with $L(w) \in I_{\rho(G)}$. Since G is k-regular, Lemma 3.1 implies that for every label $l \in I_j \neq I_{\rho(G)}$, w is adjacent to some vertex labeled l. Hence, $\sum_{i=0}^{\rho(G)-1} |I_i| = k$. By similar consideration of a vertex v with $L(v) \in I_0$, we have $\sum_{i=1}^{\rho(G)} |I_i| = k$. Thus, by the two summations, $|I_0| = |I_{\rho(G)}| = 2$.

Since $|I_j| = 2$ for all j, $0 \leq j \leq \rho(G)$, we have $I_j = \{3j, 3j+1\}$. Hence, $\lambda(G) = 3\rho(G) + 1$. But as indicated above, for v a vertex with $L(v) = 0$, v has neighbors with labels precisely the elements of $\bigcup_{i=1}^{\rho(G)} I_i$. Hence, $k = |\bigcup_{i=1}^{\rho(G)} I_i| = 2\rho(G)$, so $\lambda(G) = \frac{3}{2}k + 1$.

Proof of (3). Since $I_{\rho(G)} \geq 3$, the label $\lambda(G) - 1$ is an interior label. Thus, for vertex v with $L(v) = \lambda(G) - 1$, the neighbors of v are assigned distinct labels not in $\{\lambda(G) - 2, \lambda(G) - 1, \lambda(G)\}$, implying that L assigns at least $d(v) + 3 = k + 3$ labels. Hence, $\lambda(G) \geq (k + 3 + \rho(G)) - 1 = k + 2 + \rho(G)$.

To show that $\rho(G) \leq k - 2$, we note by Theorem 3.4 that $\rho(G) \leq k$. Since not every island of L is an atoll, then $\rho(G) \neq k$ by Theorem 3.5. The result follows by Theorem 3.7 and the observation that $|I_{\rho(G)}| \geq 3$ implies that G cannot be a sum of 4-cycles. □

We note that K_n and the complete multipartite graphs $K_{2,2,\ldots,2}$ satisfy Theorem 3.9(1) and (2), respectively. In regard to Theorem 3.9(3), the bound $k + 2 + \rho(G)$ is not necessarily sharp. For example, we argue as follows that there is no 5-regular graph G such that $\rho(G) = 3$ and $\lambda(G) = 10$. Suppose to the contrary that such a graph exists. Let $L \in \Lambda_\rho(G)$ such that $|I_0| \leq |I_1| \leq |I_2| \leq |I_3|$. Then $|I_0| \leq 2$ (for otherwise $\lambda(G) > 10$). If $|I_0| = 2$, then $I_0 = \{0, 1\}, I_1 = \{3, 4\}, I_2 = \{6, 7\}$ and $I_3 = \{9, 10\}$. Hence, by Lemma 3.1, each vertex v has degree 6, a contradiction. Thus, $|I_0| = 1$. Noting that $|I_1| \leq 2$, if $|I_1| = 2$, then $I_0 = \{0\}, I_1 = \{2, 3\}, I_2 = \{5, 6\}$ and $I_3 = \{8, 9, 10\}$. Hence, by Lemma 3.1, each vertex v with $L(v) = 0$ has degree 6, another contradiction. Thus $|I_1| = 1$. Now, $|I_2|$ is either 1, 2, or 3. If $|I_2| = 3$, then $I_0 = \{0\}, I_1 = \{2\}, I_2 = \{4, 5, 6\}$, and $I_3 = \{8, 9, 10\}$. Thus, by Lemma 3.1, each vertex v with $L(v) = 0$ has neighbors with labels 2, 4, 6, 8, and 10. But by the distance 1 condition and the 5-regularity of G, each vertex w with $L(w) = 9$ has a neighbor with label 0, a contradiction. A similar argument which focuses on vertices with labels 0 and 8 demonstrates that $|I_2|$ cannot be 2. Hence, $|I_2| = 1$. In this case, we have $I_0 = \{0\}, I_1 = \{2\}, I_2 = \{4\}$, and $I_3 = \{6, 7, 8, 9, 10\}$. So, by Lemma 3.1 and the 5-regularity of G, each vertex v with $L(v) \neq 0$ has a neighbor labeled 0. Thus, M_0 is a dominating set, and $|V(G)| = 6m_0$ (since G is 5-regular). Since $m_0 = m_{10}$ by Lemma 3.1, M_{10} is a dominating set as well. Therefore, since $M_9 \neq \phi$, there are adjacent vertices with respective labels 9 and 10, a contradiction.

We have been unable to find a 5-regular graph G with $\rho(G) = 3$. We conjecture that if G is a k-regular graph with $\rho(G) \geq 1$, then $\rho(G)$ divides k.

4. Relating $\rho(G)$, $\lambda(G)$, and $\mu(G)$. For purposes of this discussion, it will be convenient to consider the two cases $\lambda(G) \geq n-1$ and $\lambda(G) \leq n-2$, where $n = |V(G)|$. We begin with the case $\lambda(G) \geq n-1$.

THEOREM 4.1. *Let G be a graph with order n and $\lambda(G) \geq n-1$. Then*

1. $\rho(G) = c(G^c) - 1 = \lambda(G) - (n-1)$, *and*
2. *for* $L \in \Lambda_\rho(G)$, $m_i(G, L) = 0$ *or* 1.

Proof of (1). Since $\lambda(G) \geq n-1$, it follows from Theorem 2.5 that $c(G^c) - 1 = \lambda(G) - (n-1)$.

Let $\mathcal{C}$ be a path covering of G^c with minimum order. Then $\mathcal{C}$ induces a λ-labeling of G with $c(G^c) - 1$ holes (see [12]). Hence, $\rho(G) \leq c(G^c) - 1 = \lambda(G) - (n-1)$.

Now let $L \in \Lambda_\rho(G)$ and let $H(L)$ and $N(L)$ denote the set of holes of L and the set of labels assigned by L, respectively. We observe that $|H(L)| = \rho(G)$ and that $|H(L)|+|N(L)|-1 = \lambda(G)$. Thus, $\lambda(G) = (n-1)+(c(G^c)-1) = |H(L)|+|N(L)|-1 = \rho(G) + |N(L)| - 1 \leq \rho(G) + n - 1$, giving $\rho(G) \geq \lambda(G) - (n-1)$.

Proof of (2). Select $L \in \Lambda_\rho(G)$. We have seen $\lambda(G) = n + c(G^c) - 2 = |N(L)| + \rho(G) - 1$. It thus follows that $n = |N(L)|$ by (1). □

COROLLARY 4.2. *Let G be a graph with order n and $\lambda(G) \geq n-1$. Then*

1. $c(G^c) \leq \Delta(G) + 1$, *and*
2. $\rho(G) \leq \chi(G) - 1$.

Proof.

1. By Theorems 4.1 and 3.4, $c(G^c) - 1 = \rho(G) \leq \Delta(G)$.
2. For any graph G, $c(G^c) \leq \chi(G)$. The result follows by Theorem 4.1. □

We now turn our attention to graphs G with $\lambda(G) \leq n-2$, and consider the upper bound on the invariant $\mu(G)$ given by Fishburn and Roberts in the following theorem.

THEOREM 4.3. *See* 7. *If G is a graph such that $\rho(G) \geq 1$ and $\lambda(G) \leq n-2$, then $\mu(G) \leq \lambda(G) + \rho(G)$.*

It is easily seen that for $\rho(G) \geq 1$, a lower bound for $\mu(G)$ is $\lambda(G) + 1$. Thus by Theorem 4.3, $\mu(G) = \lambda(G) + 1$ if $\rho(G) = 1$. It is also immediate from Theorem 3.4 that an alternative upper bound for $\mu(G)$ is $\lambda(G) + \Delta(G)$.

We now improve the upper bound of $\lambda(G) + \rho(G)$ in the cases $\rho(G) = \Delta(G) - 1$ and $\rho(G) = \Delta(G)$.

THEOREM 4.4. *Suppose G is a graph with order n, $\lambda(G) \leq n-2$, and $\rho(G) = \Delta(G) \geq 1$. Then $\mu(G) = \lambda(G) + 1$.*

Proof. By Theorem 3.5, G is Δ-regular with $\lambda(G) = 2\Delta$, and for each L in $\Lambda_\rho(G)$, L induces $\Delta + 1$ islands $I_0, I_1, \ldots, I_\Delta$, where I_i is the atoll $\{2i\}$. By Lemma 3.1 and Theorem 3.5(3), then $n = m_0(\Delta + 1)$, implying $2\Delta \leq m_0(\Delta + 1) - 2$. This gives $m_0 \geq 2$.

By Lemma 3.1, we may denote the m_0 elements of M_{2i} by $v_{1,2i}, v_{2,2i}, \ldots, v_{m_0,2i}$ where, with no loss of generality, $v_{j,2i}$ is adjacent to $v_{j,2i+2}$. In particular, with j fixed equal to 1, $v_{1,0}, v_{1,2}, v_{1,4}, \ldots, v_{1,2\Delta}$ is a path in G. It now suffices to produce a no-hole $L(2,1)$-labeling L^* of G with span $2\Delta + 1 = \lambda(G) + 1$, which we do as follows:

$$L^*(v) = \begin{cases} L(v) + 1 & \text{if } v = v_{1,2i} \text{ for some } i, \\ L(v) & \text{otherwise.} \end{cases} \quad \square$$

THEOREM 4.5. *Suppose G is a graph with order n, $\lambda(G) \leq n-2$, and $\rho(G) = \Delta(G) - 1$. Then*

1. $\mu(G) = \lambda(G)$ *if* $\Delta(G) = 1$;
2. $\mu(G) = \lambda(G) + 1$ *if* $\Delta(G) \geq 2$.

Proof. By Theorem 3.6, $2\Delta(G) - 1 \leq \lambda(G) \leq 2\Delta(G)$. We first consider the case $\lambda(G) = 2\Delta(G)$.

Case 1: $\lambda(G) = 2\Delta(G)$.

If $\Delta(G) = 1$, then $\rho(G) = 0$, implying $\mu(G) = \lambda(G)$.

If $\Delta(G) = 2$, then by Theorem 3.6(2), G is isomorphic to either mC_4 (for some positive integer m) or $H + K_1$ where $\rho(H) = \Delta(G) = 2$. In the former case, $\lambda(G) = 4 \leq n-2 = 4m-2$, implying $m \geq 2$. By labeling the vertices of $m-1$ copies of C_4 with integers 0, 3, 1, 4, and labeling the vertices in the remaining copy of C_4 with integers 1, 4, 2, 5, we produce a no-hole $L(2,1)$-labeling of H with span $5 = \lambda(G)+1$. Thus, there exists a no-hole labeling of G with span $\lambda(G) + 1$ as well. But $\rho(G) = \Delta(G) - 1 = 1$, so $\mu(G) > \lambda(G)$. This implies $\mu(G) = \lambda(G) + 1$. In the latter case, Fishburn and Roberts [6] show that H is necessarily isomorphic to $mC_3 + kC_6$ for some integers $m, k \geq 0$. Since $4 = \lambda(G) \leq n - 2 = (3m + 6k + 1) - 2$, it follows that $m \geq 2$ or $k \geq 1$. In either event, it is easy to establish a no-hole $L(2,1)$-labeling of H with span $5 = \lambda(G) + 1$, from which it follows as above that $\mu(G) = \lambda(G) + 1$.

If $\Delta(G) \geq 3$, then by Theorem 3.6(3), G is isomorphic $H + K_1$ where $\rho(H) = \Delta(G)$. But $\Delta(G) = \Delta(H)$, so by Theorem 3.5, $\lambda(H) = 2\Delta(H)$ and $|V(H)| = w(\Delta(H) + 1)$ for some $w \geq 1$. Hence, since $\lambda(H) = \lambda(G) \leq n - 2$, we have $2\Delta(H) \leq n - 2 = |V(H)| + 1 - 2 = w(\Delta(H) + 1) - 1$, implying $w \geq 2$. This implies $\lambda(H) \leq |V(H)| - 2$. By Theorem 4.4, $\mu(H) = \lambda(H) + 1 = \lambda(G) + 1$, which implies that H (and therefore G) have no-hole labelings with span $\lambda(G) + 1$. But $\rho(G) = \Delta(G) - 1 > 1$, so $\mu(G) > \lambda(G)$. Thus $\mu(G) = \lambda(G) + 1$.

We now turn to the case $\lambda(G) = 2\Delta(G) - 1$. Let $L \in \Lambda_\rho(G)$, where $|I_0| \leq |I_1| \leq \cdots \leq |I_\rho|$. Then $I_j = \{2j\}$ for $0 \leq j \leq \rho - 1$ and $I_\rho = \{2\rho, 2\rho + 1\} = \{2\Delta(G) - 2, 2\Delta(G) - 1\}$. Hence, L assigns $\rho(G) + 2 = \Delta(G) + 1$ distinct labels, each of which is coastal. By Lemma 3.1, $m_i = m_0$ for every label i assigned by L. Therefore $n = m_0(\Delta(G)+1)$, giving $\lambda(G) = 2\Delta(G)-1 \leq n-2 = m_0(\Delta(G)+1)-2$, which implies $m_0 \geq 2$. For $0 \leq i \leq \Delta(G)-1$, let $M_{2i} = \{v_{1,2i}, v_{2,2i}, \ldots, v_{m_0,2i}\}$. By Lemma 3.1 and without loss of generality, we may suppose $v_{j,2i}$ is adjacent to $v_{j,2i+2}$, $1 \leq j \leq m_0$, $0 \leq i \leq \Delta(G) - 2$. In particular, with j fixed equal to 1, $v_{1,0}, v_{1,2}, v_{1,4}, \ldots, v_{1,2\Delta(G)-2}$ is a path in G. It now suffices to produce a no-hole $L(2,1)$-labeling L^* of G with span $\lambda(G) + 1 = 2\Delta(G)$, which we perform as follows:

$$L^*(v) = \begin{cases} L(v) & \text{if } v = v_{1,2i} \text{ for some } i, 0 \leq i \leq \Delta(G) - 1, \\ L(v) + 1 & \text{otherwise.} \end{cases} \qquad \square$$

5. On the structure of graphs G with $\rho(G) = \Delta(G)$. As shown in Theorem 3.5, for each graph G with $\rho(G) = \Delta(G)$ and each $L \in \Lambda_\rho(G)$,

1. G is Δ-regular with $|V(G)| \equiv 0$ *mod* $(\Delta(G) + 1)$;
2. $\lambda(G) = 2\Delta(G)$;
3. $M_{2j}(G, L)$ is a dominating set for each j, $0 \leq j \leq \Delta(G)$;
4. $I_j = \{2j\}$ for each j, $0 \leq j \leq \Delta(G)$.

Let $\mathcal{G}_{\Delta,t}$ be the collection of connected graphs G with $\rho(G) = \Delta(G) = \Delta$ and order $t(\Delta + 1)$ (implying $m_{2j}(G, L) = t$ for every $L \in \Lambda_\rho(G)$ and each j, $0 \leq j \leq \Delta(G)$.) Let $\mathcal{B}_{\Delta,t}$ be the subcollection of graphs in $\mathcal{G}_{\Delta,t}$ which are bipartite. We note that $\mathcal{G}_{\Delta,1} = \{K_{\Delta+1}\}$. We thus restrict our attention to the case $t \geq 2$, with particular emphasis on $t = 2$.

In [7], Fishburn and Roberts construct connected graphs G with $\lambda(G) = 2m$, $|V(G)| = 2(m+1)$, and $\rho(G) = m$, for $m \geq 2$. We note that for $m = 2$, the constructed graph is isomorphic to C_6, and for $m \geq 3$, the constructed graph is not bipartite. Thus, it follows that for $\Delta \geq 2$, $\mathcal{B}_{2,2}$, and $\mathcal{G}_{\Delta,2}$ are not empty. We also note that $\mathcal{B}_{2,2} = \mathcal{G}_{2,2}$.

The following lemma will assist in characterizing $\mathcal{B}_{\Delta,2}$ for all $\Delta \geq 2$.

LEMMA 5.1. *If G is a connected Δ-regular graph of order $2(\Delta+1)$, then $G \in \mathcal{G}_{\Delta,2}$ or $\lambda(G) = 2\Delta + 1$.*

Proof. Since G^c is a $(\Delta+1)$-regular graph on $2(\Delta+1)$ vertices, then by Dirac's theorem [5], G^c has a Hamilton path. Hence, by Theorem 2.5, $\lambda(G) \leq |V(G)| - 1 = 2\Delta + 1$. It suffices to show that if $\lambda(G) \leq 2\Delta$, then $G \in \mathcal{G}_{\Delta,2}$.

Let L be an arbitrary $L(2,1)$-labeling of G with span $s(L)$, $\lambda(G) \leq s(L) \leq 2\Delta$. If v and w are vertices in $V(G)$ such that $L(v) = L(w) = l$, then $\{v, w\}$ is a dominating set due to the distance conditions and regularity and order of G. Hence, there exists no vertex with label $l-1$ or $l+1$, which in turn implies $m_i + m_{i+1} \leq 2$ for each i, $0 \leq i \leq s(L) - 1$. Therefore, $|V(G)| = 2\Delta + 2 \leq 2\lfloor \frac{s(L)+2}{2} \rfloor$, giving $s(L) \geq 2\Delta$. Since L was arbitrary, $\lambda(G) \geq 2\Delta$ as well, giving $\lambda(G) = 2\Delta$.

Now let L be an arbitrary λ-labeling of G. To see that L necessarily has Δ holes, we note that since $\lambda(G) = 2\Delta$, then $|V(G)| = 2\Delta+2 = (m_0+m_1)+(m_2+m_3)+\cdots+(m_{2\Delta-2}+m_{2\Delta-1})+m_{2\Delta} = m_0+(m_1+m_2)+(m_3+m_4)+\cdots+(m_{2\Delta-1}+m_{2\Delta})$. Since $m_i+m_{i+1} \leq 2$ as above, then $m_i+m_{i+1} = 2$ for all i, $0 \leq i \leq 2\Delta-1$, and $m_0, m_\Delta = 2$ as well. Hence, for $0 \leq i \leq 2\Delta - 2$, $(m_{i+2} + m_{i+1}) - (m_{i+1} + m_i) = m_{i+2} - m_i = 0$, which gives $m_i = 2$ for even i and $m_i = 0$ for odd i. ∎

Now, for $\Delta \geq 2$, let ${}_\Delta B$ be a connected Δ-regular bipartite graph with order $2(\Delta+1)$. It is easy to see that ${}_\Delta B$ can be obtained by deleting a perfect matching from $K_{\Delta+1,\Delta+1}$, and is unique up to isomorphism.

THEOREM 5.2. *For $\Delta \geq 2$, $\mathcal{B}_{\Delta,2} = \{{}_\Delta B\}$.*

Proof. Since ${}_\Delta B$ has diameter 3, then for every vertex $v \in V({}_\Delta B)$, there exists a unique vertex $w \in V({}_\Delta B)$ such that $d(v,w) = 3$. Hence there exists an $L(2,1)$-labeling of ${}_\Delta B$ with span 2Δ. Thus, by Lemma 5.1, ${}_\Delta B \in \mathcal{G}_{\Delta,2}$, implying ${}_\Delta B \in \mathcal{B}_{\Delta,2}$. ∎

From Theorem 5.2 and the discussion preceding Lemma 5.1, it follows that $|\mathcal{G}_{m,2}| \geq 2$ for $m \geq 3$. We further note that $\mathcal{B}_{3,2} = \{Q_3\}$.

To determine $\mathcal{G}_{3,2}$, we consider the four nonisomorphic connected 3-regular graphs of order 8 (see [1]) as shown in Figure 5.1.

The graph in Figure 5.1(a) is the graph constructed by Fishburn and Roberts, while the graph in Figure 5.1(b) is Q_3. Each is clearly in $\mathcal{G}_{3,2}$. On the other hand, if $G \in \mathcal{G}_{\Delta,2}$, then $V(G)$ can be partitioned into $\Delta(G)+1$ sets containing precisely 2 vertices which are exactly distance 3 apart. Since the diameter of the graph in Figure 5.1(d) is 2, its λ-number is 7 by Lemma 5.1. And since, in Figure 5.1(c), there is a vertex which is at most distance 2 from every other vertex, that graph is not in $\mathcal{G}_{3,2}$. It follows from Lemma 5.1 that the λ-number of this graph is 7 as well.

We next introduce a particular graph construction which will aid in characterizing $\mathcal{G}_{\Delta,2}$.

5.1. The S-exchange of the sum of two graphs. Let G be a graph with $V(G) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and for $i = 1, 2$, let ϕ_i be a graph isomorphism from G to graph G_i where $\phi_i(v_j) = v_{j,i}$. Let $e = \{v_r, v_s\} \in E(G)$. Then the e-exchange of graph $G_1 + G_2$, denoted $X_e(G_1 + G_2)$, is the graph with vertex set $V(G_1 + G_2)$ and edge set

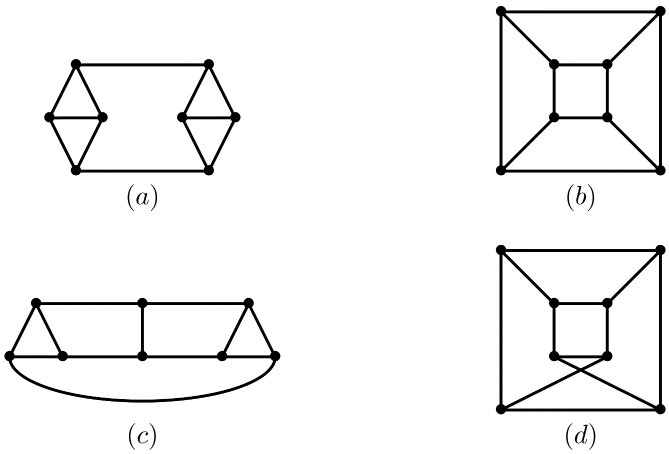

FIG. 5.1. *Four nonisomorphic connected* 3*-regular graphs of order* 8.

$(E(G_1+G_2)-\{\phi_1(e),\phi_2(e)\})\bigcup T(e)$, where $T(e)=\{\{v_{r,1},v_{s,2}\},\{v_{r,2},v_{s,1}\}\}$. Furthermore, if $S\subseteq E(G)$, then the S-exchange of graph G_1+G_2, denoted $X_S(G_1+G_2)$, is the graph with vertex set $V(G_1+G_2)$ and edge set $(E(G_1+G_2)-\bigcup_{e\in S}\{\phi_1(e),\phi_2(e)\})\bigcup(\bigcup_{e\in S}T(e))$.

By way of illustration, we note that if G is isomorphic to K_3 and $S=E(G)$, then $X_S(G_1+G_2)$ is isomorphic to C_6. Additionally, if G is isomorphic to K_4 and e is any edge in $E(G)$, then $X_e(G_1+G_2)$ is isomorphic to the graph in Figure 5.1(a). We also note that for any $v\in V(G)$, if $S(v)=\{e\in E(G)|e$ is incident to $v\}$, then $X_{S(v)}(G_1+G_2)$ is isomorphic to G_1+G_2.

THEOREM 5.3. *Let H be a connected Δ-regular graph with order $2(\Delta+1)$. Then $H\in\mathcal{G}_{\Delta,2}$ if and only if there exists $S\subseteq E(K_{\Delta+1})$ such that H is isomorphic to $X_S(K_{\Delta+1}+K_{\Delta+1})$.*

Proof. ($\Rightarrow$). Let $H\in\mathcal{G}_{\Delta,2}$ and let L be a λ-labeling of H. Then for $0\le i\le 2\Delta$, $m_i=0$ if i is odd and $m_i=2$ if i is even. Let $v_{0,1}$ and $v_{0,2}$ denote the two vertices in $V(H)$ with label 0 under L. For $i=1,2$ and for $1\le j\le\Delta$, let $v_{j,i}$ be the vertex in $V(H)$ which has label $2j$ and which is adjacent to $v_{0,i}$. Also let H_i be the subgraph of H induced by $\{v_{0,i},v_{1,i},\ldots,v_{\Delta,i}\}$ and let W be the edge set of H_1^c. (We note that H_1 is isomorphic to H_2.) Setting $S=\phi_1^{-1}(W)$ (where ϕ_1 is the graph isomorphism from G to G_1 such that $\phi(v_i)=v_{i,1}$, where $G=K_{\Delta+1}$ and $V(G)=\{v_0,v_1,\ldots,v_\Delta\}$), we easily see that H is isomorphic to $X_S(K_{\Delta+1}+K_{\Delta+1})$.

($\Leftarrow$). Suppose $S\subseteq E(K_{\Delta+1})$ such that H is isomorphic to $X_S(K_{\Delta+1}+K_{\Delta+1})$. Let L be the $L(2,1)$-labeling of H such that $L(v_{j,i})=2j$ for $i=1,2$. Since the span of L is $2\Delta<2\Delta+1$, Lemma 5.1 implies that $H\in\mathcal{G}_{\Delta,2}$. □

It is easily seen that the graphs in Figures 5.1(a) and 5.1(b) are S-exchanges of K_4+K_4, where, in the latter case, $|S|=2$ (for independent edges) and in the former case, $|S|=1$.

To this point, we have restricted our attention to elements of $\mathcal{G}_{\Delta,t}$ for $t=2$. Using two new graph constructions, we next extend the discussion to $2<t\le\Delta(G)$.

The graph Ω_r. For $r\ge 1$, let $X=rK_r$ and $Y=rK_1$. We form a new graph Ω_r by joining the vertices of Y to certain vertices of X. Formally, let $V(\Omega_r)=V(X)\bigcup V(Y)$ where

1. $V(X)=\bigcup_{i=0}^{r-1}B_i$, $B_i=\{b_{i,j}|0\le j\le r-1\}$, and

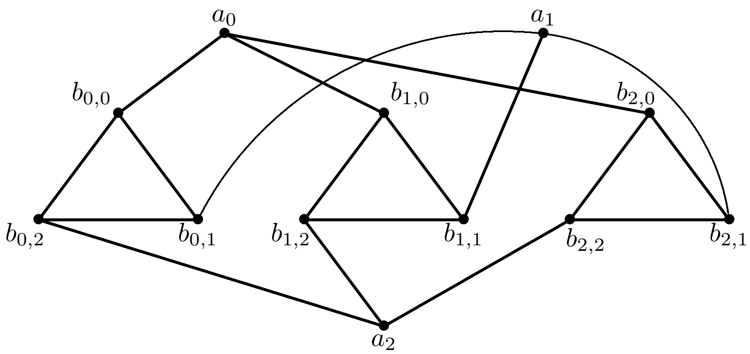

FIG. 5.2. *The graph* Ω_3.

2. $V(Y) = \{a_0, a_1, ..., a_{r-1}\}$.

Let $E(\Omega_r) = R \bigcup S$, where

3. $R = \bigcup_{i=0}^{r-1} R_i$, where $R_i = \{\{b_{i,j}, b_{i,k}\} | 0 \leq j < k \leq r-1\}$, and
4. $S = \bigcup_{i=0}^{r-1} S_i$, where $S_i = \{\{a_i, b_{m,i}\} | 0 \leq m \leq r-1\}$.

We note that Ω_1 is isomorphic to K_2, and Ω_2 is isomorphic to C_6. We illustrate Ω_3 in Figure 5.2.

We make the following observations about the structure of Ω_r:

Obs. 1) Ω_r is r-regular and has order $r^2 + r$; $|V(X)| = r^2$ and $|V(Y)| = r$;

Obs. 2) for $0 \leq i, j \leq r-1$, $d(a_j, a_i) = 3$ for $j \neq i$;

Obs. 3) for $0 \leq i, j, k, l \leq r-1$

$$d(b_{i,j}, b_{k,l}) = \begin{cases} 1 & \text{if } i = k \text{ and } j \neq l, \\ 2 & \text{if } i \neq k \text{ and } j = l, \\ 3 & \text{otherwise;} \end{cases}$$

Obs. 4) For $0 \leq i, j, k \leq r-1$,

$$d(a_i, b_{j,k}) = \begin{cases} 1 & \text{if } i = k, \\ 2 & \text{otherwise.} \end{cases}$$

LEMMA 5.4. *Let L be an $L(2,1)$-labeling of Ω_r. Then*

1. *for every $y \in V(Y)$ and every $x \in V(X)$, $L(x) \neq L(y)$;*
2. *for $0 \leq t \leq s(L) - 1$, $m_t + m_{t+1} \leq r$.*

Proof. By Obs. 4, (1) follows.

To show (2), suppose to the contrary that there exists t, $0 \leq t \leq s(L) - 1$, such that $m_t + m_{t+1} \geq r + 1$. From Obs. 2, 3, 4, either every vertex labeled t (resp. $t+1$) is in $V(X)$ or every vertex labeled t (resp. $t+1$) is in $V(Y)$. Furthermore, if every vertex in $M_t \bigcup M_{t+1}$ is in $V(Y)$, then we have the contradiction that $r + 1 \leq m_t + m_{t+1} \leq |V(Y)| = r$. Similarly, if every vertex in $M_t \bigcup M_{t+1}$ is in $V(X)$, then by the pigeon-hole principle, there exist two vertices $b_{i,j}, b_{k,l}$ in $M_t \bigcup M_{t+1}$ where $i = k$. Thus, $b_{i,j}$ and $b_{k,l}$ are adjacent, a contradiction of the assumption that their labels under L differ by at most 1. We have therefore established that either $M_t \subseteq V(Y)$ and $M_{t+1} \subseteq V(X)$ or $M_t \subseteq V(X)$ and $M_{t+1} \subseteq V(Y)$.

Suppose the former. Let $s_t = \{i | a_i \in M_t\}$ and let $s_{t+1} = \{k | b_{j,k} \in M_{t+1}$ for some $j\}$. We observe that $|s_t| = m_t$, and from Obs. 3, $|s_{t+1}| = m_{t+1}$. Noting that s_t and s_{t+1} are subsets of $\{0, 1, 2, \ldots, r-1\}$, $|s_t| + |s_{t+1}| = m_t + m_{t+1} \geq r + 1$ implies

$s_t \bigcap s_{t+1} \neq \phi$. Thus, for some integers y, z, $0 \leq y, z \leq r-1$, there exist adjacent vertices a_y and $b_{z,y}$ in $M_t \bigcup M_{t+1}$, a contradiction of the distance one condition on L.

A similar argument can be made in the latter case. □

THEOREM 5.5. *For $r \geq 1$, $\Omega_r \in \mathcal{G}_{r,r}$.*

Proof. We first establish that $\lambda(\Omega_r) = 2r$. Suppose $\lambda(\Omega_r) < 2r$. Let L be an $L(2,1)$-labeling of Ω_r with span $2r-1$. By Obs. 1, $r^2 + r = |V(\Omega_r)| = \sum_{i=0}^{2r-1} m_i$. However, by Lemma 5.4, $\sum_{i=0}^{2r-1} m_i = \sum_{j=0}^{r-1}(m_{2j} + m_{2j+1}) \leq r^2$, a contradiction. Hence, $\lambda(\Omega_r) \geq 2r$. To show that $\lambda(\Omega_r) = 2r$, let $B_k = \{b_{i,j} | (j-i) \equiv k \bmod r\}$, $0 \leq k \leq r-1$. Noting that $|B_k| = r$ and that vertices in B_k are pairwise distance 3 apart, we produce an $L(2,1)$-labeling L of Ω_r as follows:

$$L(v) = \begin{cases} 2k & \text{if } v \in B_k, \\ 2r & \text{otherwise.} \end{cases}$$

To show $\rho(\Omega_r) = r$, let L^* be any λ-labeling of Ω_r. Then $r^2 + r = \sum_{i=0}^{2r} m_i = m_{2r} + \sum_{i=0}^{2r-1} m_i$. By Lemma 5.4, $\sum_{i=0}^{2r-1} m_i \leq r^2$, implying $m_{2r} = r$. By Obs. 1 (the r-regularity of Ω_r in particular), M_{2r} is therefore a dominating set. Thus, $m_{2r-1} = 0$. Proceeding by induction, it is easily seen that for $0 \leq j \leq 2r$,

$$m_j = \begin{cases} r & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Hence, $\rho(\Omega_r) = r$. □

Theorem 5.5 establishes the fact that $\mathcal{G}_{r,r}$ is nonempty. Earlier discussions have demonstrated that $\mathcal{G}_{r,1} = \{K_{r+1}\}$, and that for $r \geq 2$, $\mathcal{G}_{r,2}$ is nonempty. The question is thus raised: for what values of t is $\mathcal{G}_{r,t}$ nonempty?

To see that such graphs exist for arbitrary $t < r$, we introduce one last graph construction.

The graph $\boldsymbol{\Omega_{r,t}}$. Fix integers t and r such that $1 \leq t \leq r$. Let $X = tK_r$ and let $Y = tK_1$. We form a new graph $\Omega_{r,t}$ by joining the vertices in Y to certain vertices in X. Formally, let $V(\Omega_{r,t})$ equal $V(X) \bigcup V(Y)$, where

1. $V(X) = \bigcup_{i=0}^{t-1} B_i$, $B_i = \{b_{i,j} | 0 \leq j \leq r-1\}$, and
2. $V(Y) = \{a_0, a_1, ..., a_{t-1}\}$.

Let $E(\Omega_{r,t}) = R \bigcup S \bigcup T$, where

3. $R = \bigcup_{i=0}^{t-1} R_i$ where $R_i = \{\{b_{i,j}, b_{i,k}\} | 0 \leq j < k \leq r-1\}$, and
4. $S = \bigcup_{i=0}^{t-1} S_i$, where $S_i = \{\{a_i, b_{m,i}\} | 0 \leq m \leq t-1\}$, and
5. $T = \bigcup_{i=0}^{t-1} T_i$, where $T_i = \{\{a_i, b_{i,j}\} | t \leq j \leq r-1\}$.

We illustrate $\Omega_{4,2}$ in Figure 5.3.

We note that $\Omega_{2,1}$ is isomorphic to K_3, and in general $\Omega_{r,1}$ is isomorphic to K_{r+1}. We also note that $\Omega_r = \Omega_{r,r}$, and that $\Omega_{3,2}$ is isomorphic to the graph in Figure 5.1(a).

Arguments similar to those used in the analysis of Ω_r demonstrate that $\Omega_{r,t}$ is a graph G with $\rho(G) = r$ and $m_{2i}(G, L) = t$ for $L \in \Lambda_\rho(G)$.

We observe that the edges of $\Omega_{r,t}$ may be manipulated to produce other graphs G with $\rho(G) = r$ and $m_i(G, L) = t$ for $L \in \Lambda_\rho(G)$. Such a graph is illustrated in Figure 5.4 for $r = 4, t = 2$.

We point out that the graphs in Figures 5.3 and 5.4 can be constructed as S-exchanges of $K_5 + K_5$.

We have been unable to establish that $\mathcal{G}_{r,t}$ is nonempty for $t > r$, and conjecture that $\mathcal{G}_{r,t} = \phi$ for all $t > r$.

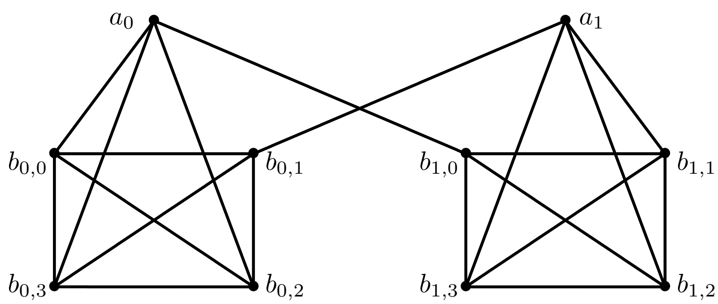

FIG. 5.3. *The graph* $\Omega_{4,2}$.

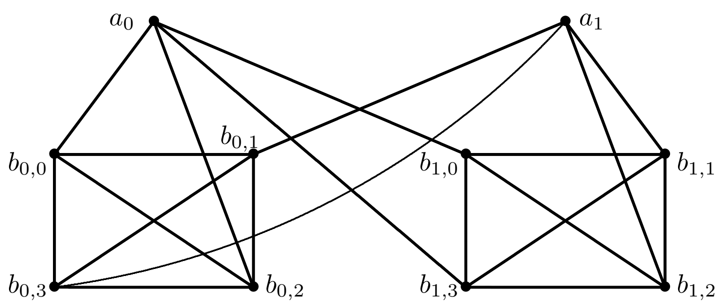

FIG. 5.4. *Graph* G *with* $\rho(G) = 4$ *and* $m_i(G, L) = 0$ *or* 2 *for* $L \in \Lambda(G, \rho)$.

6. Closing remarks. We have offered several conjectures about the structure of nonfull colorable graphs in earlier sections of this paper. Throughout our investigations of graphs G with positive $\rho(G)$ we found none with $\lambda(G) > 2\Delta(G)$. Thus, we conjecture that if $\lambda(G) > 2\Delta(G)$, then $\rho(G) = 0$.

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